On the Arithmetics of Discrete Figures

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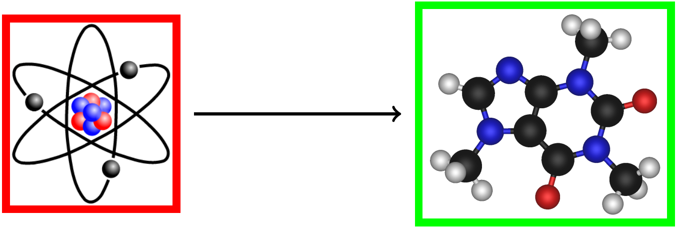
Outline

Prime and composed objects

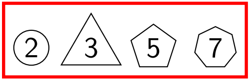
The concept of **prime object** arise naturally in chemistry...



Atom

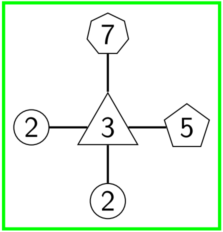
Molecule

...But also in mathematics!

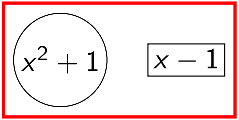


Prime numbers

$2^2 \cdot 3 \cdot 5 \cdot 7 = 420$ →

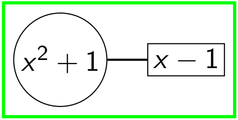


Composed number



Prime polynomials over $\mathbb{Z}$

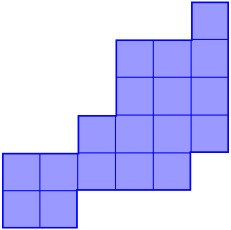
$(x^2 + 1)(x - 1)$ →
$= x^3 - x^2 + x - 1$



Composed polynomial

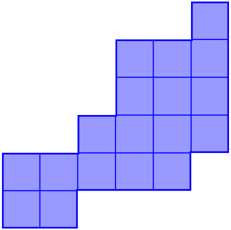
Aim of the talk

Question: What can be said about the primality of polyominoes?



Aim of the talk

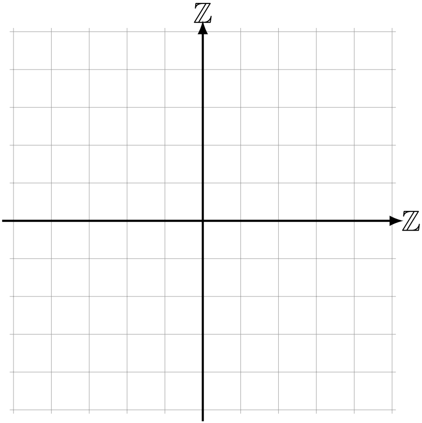
Question: What can be said about the primality of polyominoes?



Goal: Study factorization of polyominoes as a product of prime ones.

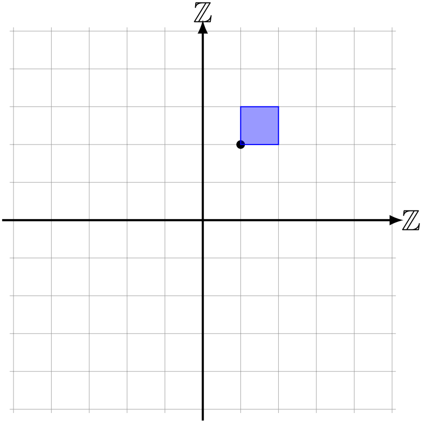
Outline

Digital geometry

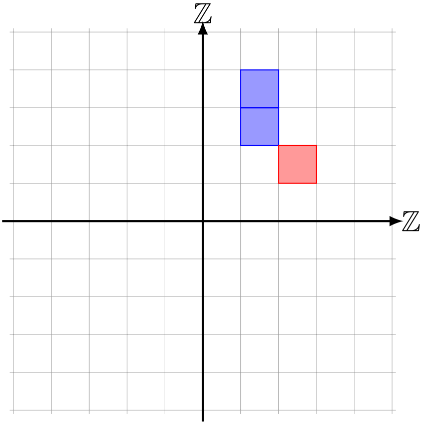


The **square grid** $\mathbb{Z} \times \mathbb{Z}$.

Digital geometry



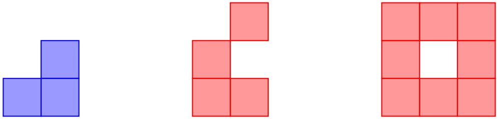
The **cell** $c(1, 2)$.



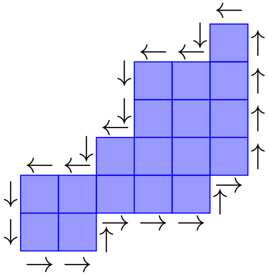
Cells $c(1,2)$ and $c(1,3)$ are 4-**connected** while cells $c(1,2)$ and $c(2,1)$ are 8-**connected**.

Polyominoes

A **polyomino** P is a 4-connected set of pixels without hole.

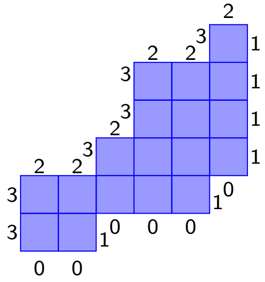


We represent polyominoes by their boundary using the four elementary steps $\rightarrow$ (east), $\uparrow$ (north), $\leftarrow$ (west) and $\downarrow$ (south).



Encoding of polyominoes

We represent polyominoes by their boundary using the four elementary steps $\rightarrow$ (0), $\uparrow$ (1), $\leftarrow$ (2) and $\downarrow$ (3).



P is represented by the contour word $w = 0010001011112322332232233$.

Combinatorics on words

An **alphabet** is a set of letters:

$$\mathcal{F} = \{0, 1, 2, 3\}$$

A **word** is a sequence of letters:

$$w = 001000101111232233232233$$

The **length** of w is denoted by $|w|$:

$$|w| = 24$$

The **free monoid** $\mathcal{F}^*$ is the set of all words over $\mathcal{F}$ with concatenation:

$$\mathcal{F}^* = \{\varepsilon, 0, 1, 2, 3, 00, 01, 02, \cdots\}$$

The **reversal of** w is the word $\widetilde{w} = w_n w_{n-1} \cdots w_1$:

$$\widetilde{w} = 332232332232111101000100$$

Combinatorics on words

Let $\mathcal{A}$ and $\mathcal{B}$ be two alphabets.
A **morphism** is a function $\varphi : \mathcal{A}^* \rightarrow \mathcal{B}^*$ such that $\varphi(uv) = \varphi(u)\varphi(v)$.

For example, the $\bar{\cdot}$ morphism:

$$\begin{array}{ccc} 0 & \longleftrightarrow & 2 \\ 1 & \longleftrightarrow & 3 \end{array}$$

An **antimorphism** is a function $\varphi : \mathcal{A}^* \rightarrow \mathcal{B}^*$ such that $\varphi(uv) = \varphi(v)\varphi(u)$.

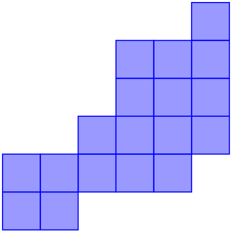
For example, the $\widehat{\cdot}$ antimorphism:

$$\widehat{\cdot} = \bar{\cdot} \circ \widetilde{\cdot}$$

Paths as words

There is a bijection between finite counter-clockwise circular boundary words on $\mathcal{F}^*$ and polyominoes.

$$w = 0010001011112322332232233 \qquad \simeq$$

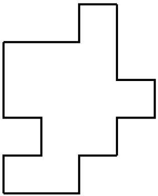


Parallelogram polyominoes

P is called **parallelogram polyomino** if and only if $P = POLY(w)$ with

$$w = xy\widehat{x}\widehat{y}.$$

For example, $w = 00101012112322330323$.



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P is called **parallelogram polyomino** if and only if $P = POLY(w)$ with

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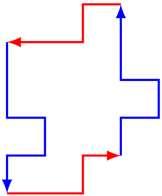
For example, $w = 0010 \cdot 101211 \cdot 2322 \cdot 330323$.



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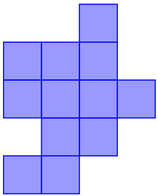

Remark that if

$$w = 0010 \cdot 101211 \cdot 2322 \cdot 330323$$

we have

$$Fst(x) = Lst(x)$$

and

$$\{Fst(x), Fst(y), Fst(\widehat{x}), Fst(\widehat{y})\} = \{0, 1, 2, 3\}$$

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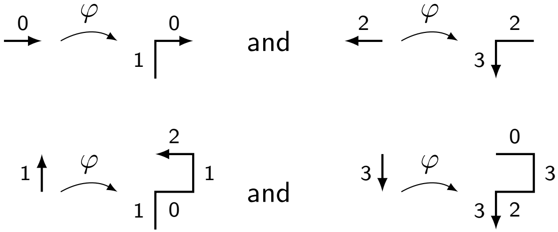
This is true in general.

Outline

Homologous morphisms

$\varphi : \mathcal{F}^* \longrightarrow \mathcal{F}^*$ is called **homologous** if $\varphi(a) = \widehat{\varphi(\overline{a})}$ for every $a \in \mathcal{F}$.
For example,

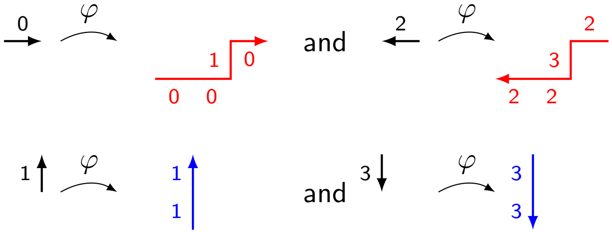


Parallelogram morphisms

An homologous morphism $\varphi : \mathcal{F}^* \longrightarrow \mathcal{F}^*$ is called **parallelogram** if

- $\varphi(0123)$ is the boundary word of a polyomino;
- $Fst(\varphi(a)) = a$ for every $a \in \mathcal{F}$.

For example,



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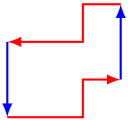

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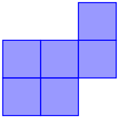
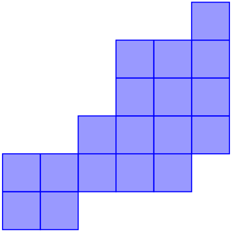


Prime polyominoes

A polyomino $P \neq$  is called **composed** if there exists some boundary word u and some parallelogram morphism φ such that

- $POLY(\varphi(u)) = P$;
- $POLY(u) \neq$ ;
- $\varphi \neq Id$.

Otherwise, it is **prime**. For example,



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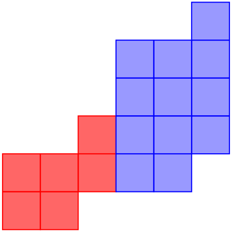

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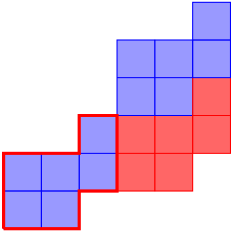

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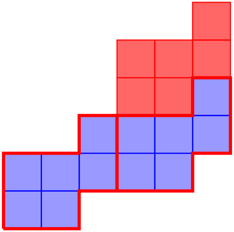

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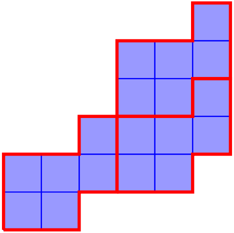

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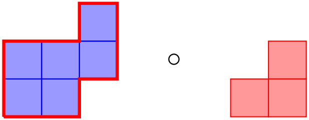

Interesting results

Theorem

If $P \neq$  *is a polyomino, then either* P *is prime or there exists prime parallelogram morphisms* $\varphi_1, \varphi_2, \ldots, \varphi_n$ *and a prime boundary word* u *such that* $P = POLY((\varphi_1 \circ \varphi_2 \circ \cdots \circ \varphi_n)(u))$.

Theorem

Let p *be a prime number. Then, any polyomino of area* p *is prime.*

Outline

Computing decomposition

By definition, given the contour word w of a polyomino P, we need to find a parallelogram morphism $\varphi \neq Id$ and a boundary word $u \neq 0123$ such that

$$\varphi(u) = w.$$

Input: A contour word w of a polyomino P.

Output: A couple (φ, u) if P is composed and `FALSE` otherwise.

Computing decomposition

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$$\varphi(u) = w.$$

Input: A contour word w of a polyomino P. $\mathcal{O}(n)$

Output: A couple (φ, u) if P is composed and `FALSE` otherwise.

Naive algorithm

Idea: Compute a factorization using the definition.

$$w = 0010001011112322233232233$$

Step 1: Compute two factors $f_0 \in 0\mathcal{F}^*0$ and $f_1 \in 1\mathcal{F}^*1$ of w;

$$f_0 = 0010 \text{ and } f_1 = 11$$

Step 2: Compute the homologous morphism φ induced by f_0 and f_1;

$$\varphi(0) = 0010,\ \varphi(1) = 11,\ \varphi(2) = \widehat{\varphi(0)} = 2322 \text{ and } \varphi(3) = \widehat{\varphi(1)} = 33$$

Step 3: If there exists $u \neq 0123$ such that $\varphi(u) = w$, return (φ,u);

We have $u = 00112323$ and
$$\varphi(u) = 0010 \cdot 0010 \cdot 11 \cdot 11 \cdot 2322 \cdot 33 \cdot 2322 \cdot 33 = w$$

Step 4: Otherwise, repeat steps $1 - 3$ for all remaining f_0 and f_1.

Naive algorithm

Idea: Compute a factorization using the definition. $\mathcal{O}(n^6)$

$$w = 00100010111123223323223 3$$

Step 1: Compute two factors $f_0 \in 0\mathcal{F}^*0$ and $f_1 \in 1\mathcal{F}^*1$ of w;

$$f_0 = 0010 \text{ and } f_1 = 11$$

Step 2: Compute the homologous morphism φ induced by f_0 and f_1; $\mathcal{O}(1)$

$$\varphi(0) = 0010,\ \varphi(1) = 11,\ \varphi(2) = \widehat{\varphi(0)} = 2322 \text{ and } \varphi(3) = \widehat{\varphi(1)} = 33$$

Step 3: If there exists $u \neq 0123$ such that $\varphi(u) = w$, return (φ,u); $\mathcal{O}(n)$

We have $u = 00112323$ and

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Step 4: Otherwise, repeat steps $1 - 3$ for all remaining f_0 and f_1. $\mathcal{O}(n^4)$

Improving the naive algorithm

Idea: Be more sensible about the choice of factors.
Step 1: Choose a block $b_0 \in 0\mathcal{F}^*0$;

$$w = \boxed{00}\,1000101111232233232233$$

Step 2: If the next letter is 0 or 2, check if the next block matches. Else, return to **Step 1**;

$$w = \boxed{00}\,1000101111232233232233$$

Step 3: Repeat steps $1-2$ for a block $b_1 \in 1\mathcal{F}^*1$;

$$w = \boxed{00}\,\boxed{10001}\,01111232233232233$$

Step 4: Verify if w can be decomposed into blocks.

$$w = \boxed{00}\,\boxed{10001}\,\boxed{01}\,111232233232233$$

Improving the naive algorithm

Idea: Be more sensible about the choice of factors.

Step 1: Choose a block $b_0 \in 0\mathcal{F}^*0$;

$$w = \boxed{0010}\,001011112322332322 33$$

Step 2: If the next letter is 0 or 2, check if the next block matches. Else, return to **Step 1**;

$$w = \boxed{0010}\,\boxed{0010}\,111123223323 2233$$

Step 3: Repeat steps $1-2$ for a block $b_1 \in 1\mathcal{F}^*1$;

$$w = \boxed{0010}\,\boxed{0010}\,\boxed{11}\,11232233232233$$

Step 4: Verify if w can be decomposed into blocks.

$$w = \boxed{0010}\,\boxed{0010}\,\boxed{11}\,\boxed{11}\,\boxed{2322}\,\boxed{33}\,\boxed{2322}\,\boxed{33}$$

Improving the naive algorithm

Idea: Be more sensible about the choice of factors. $\mathcal{O}(n^5)$

Step 1: Choose a block $b_0 \in 0\mathcal{F}^*0$; $\mathcal{O}(n)$ choices for b_0

$w = [0010]\,001011112322332322233$

Step 2: If the next letter is 0 or 2, check if the next block matches. Else, return to **Step 1**;

$w = [0010]\,[0010]\,1111232233232233$

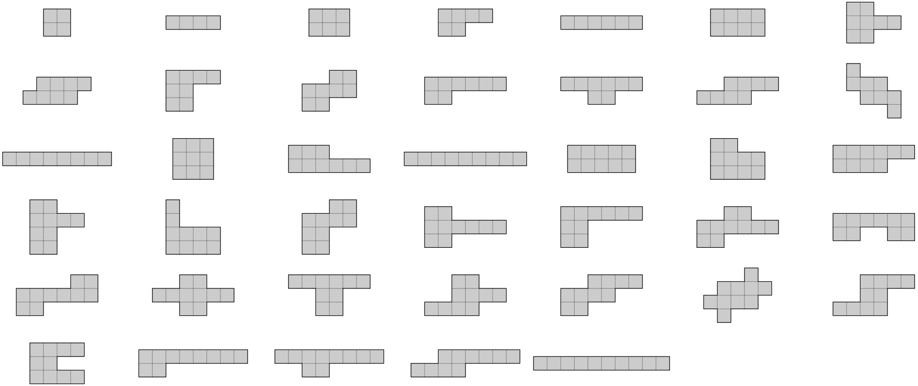
Step 3: Repeat steps $1-2$ for a block $b_1 \in 1\mathcal{F}^*1$; $\mathcal{O}(n)$ choices for b_1

$w = [0010]\,[0010]\,[11]\,11232233232233$

Step 4: Verify if w can be decomposed into blocks. $\mathcal{O}(n)$

$w = [0010]\,[0010]\,[11]\,[11]\,[2322]\,[33]\,[2322]\,[33]$

Composed polyominoes of area ≤ 10



Outline

What we know:

- Any polyomino can be decomposed into prime ones;
- This decomposition is computed in polynomial time ($\mathcal{O}(n^5)$ for the improved algorithm).

What we don't know:

- How to characterize polyominoes with unique prime decomposition? ($w = 0^m1^n2^m3^n$ has multiple decompositions);
- Could the time complexity be lowered? (probably by considering repetition of factors);
- How many prime and composed polyominoes of area n are there?

Questions?

Gracias!