Algebraic Tools for the Product of Overlapping Tiles

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Research context

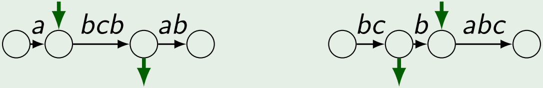
1. Development of a language theory for inverse monoids.

A monoid S is an inverse monoid when for any $x \in S$, there exists a unique x^{-1} so that $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x$.

2. Quasi-recognizable languages of tiles.
   (using MacAlister's inverse monoid)
3. Closure under product and restricted product.

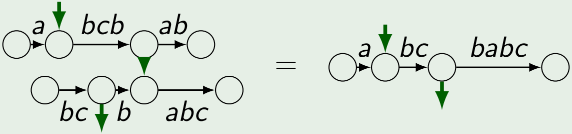
The Monoid of Tiles

Example (Tiles (a, bcb, ab) and $(bc, b, abc)^{-1} = (bcb, \bar{b}, babc)$)



$\mathcal{T}(A)$: inverse monoid of overlapping tiles (i.e. birooted words[1]): product, neutral element $1 = (1, 1, 1)$, absorbing element 0.

Example (the product $(a, bcb, ab)(bc, b, abc)^{-1}$)

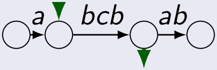


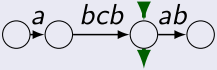
When these conditions are no met, $uv = 0$.

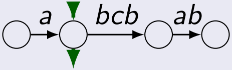
[1] D.B. McAlister, Inverse semigroups which are separated over a subsemigroup, Trans. Amer. Math. Soc., vol. 182, pp. 85-117 (1973)

Natural Order over Tiles

Definition (Left and right-projections)

For any $u =$ 

$u^L =$ 

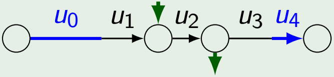
and $u^R =$ 

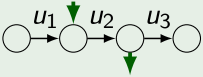
Remark $uu^L = u = u^R u$.

Definition (Natural order over tiles[1])

For any $u, v \in \mathcal{T}(A)$, $u \leq v$ when $u = vu^L$ or $u = u^R v$.

Example



$\leq$


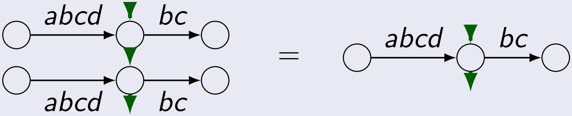
[1] K.S.S. Nambooripad, The natural partial order on a regular semigroup, Proc. Edinburgh Math. Soc., vol. 23, pp.249–260, (1983)

The monoid of Tiles: Remarkable Elements

Subunits

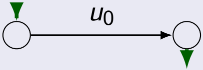
$U(\mathcal{T}(A)) = \{u \in \mathcal{T}(A) \mid u \leq 1\} = \{(u_1, u_2, u_3) \in \mathcal{T}(A) \mid u_2 = 1\}$

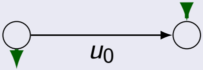
Elements of the set of subunits $U(\mathcal{T}(A))$ are **idempotents**.



Maximal elements

$A^* \simeq maximal\ positive\ tiles$; $(A^*)^{-1} \simeq maximal\ negative\ tiles$

$u_0 \simeq$ 

; $u_0^{-1} \simeq$ 

We use the embedding $u_0 \rightarrow (1, u_0, 1)$.

Left and right-projections

$u^L = min\{v \in U(\mathcal{T}(A)) \mid uv = u\}$, $u^R = min\{v \in U(\mathcal{T}(A)) \mid vu = u\}$.

Definability of Languages of Tiles

Theorem (MSO-definability)

A language of tiles is MSO-definable iff it is a finite union of languages of the form $U^L V W^R$, with U, V and W being regular languages of A^*.

Fact

Recognizability by **morphisms** in **finite monoids** collapses over tiles[1].

Definition (Quasi-recognizability)

A language $L \subseteq \mathcal{T}(A)$ is quasi-recognizable, i.e. $L \in Q\text{-}REC$, when there exists an **adequate premorphism** $\varphi : \mathcal{T}(A) \to S$, with S an **E-ordered monoid**, so that $L = \varphi^{-1}(\varphi(L))$.

[1] D. Janin, On languages of one-dimensional overlapping tiles, Int. Conf. on Current Thrends in Theo. and Prac. Comp. Science (SOFSEM). LNCS, vol. 7741, pp. 244-256

E-ordered Monoids

This definition comes from M. Lawson's and V. Gould's work over Ehresmann's inverse semigroups:

Definition

A finite monoid S equipped with a preorder $\preccurlyeq$ stable by product is an **E-ordered monoid** when

- S possesses a minimum 0.
- $\preccurlyeq$ is an order over $U(S)$, and $U(S)$ is a $\wedge$-semilattice with product as $\wedge$.
- For any $x \in S$, left and right projections x^L and x^R are defined.
- These projections are monotonic: if $x \preccurlyeq y$ then $x^R \preccurlyeq y^R$.
- Right and left semi-congruence induced by projections: $(xy)^L = (x^L y)^L$ and $(xy)^R = (xy^R)^R$.

Adequate Premorphisms

A **premorphism** is a **monotonic** mapping $\varphi : \mathcal{T}(A) \to S$ so that

- S is an E-ordered monoid,
- $\varphi(1) = 1$,
- for any $u, v \in \mathcal{T}(A)$ with $u \leq v$, $\varphi(u) \preccurlyeq \varphi(v)$,
- $\varphi(uv) \preccurlyeq \varphi(u)\varphi(v)$.

It is **adequate** when it preserves

- left and right-projections: $\varphi(u^R) = \varphi(u)^R$, $\varphi(u^L) = f(u)^L$,
- disjoint products: for $u = (u_1, u_2, 1)$ and $v = (1, v_2, v_3)$, $\varphi(uv) = \varphi(u)\varphi(v)$.

Closure properties of Quasi-recognizability

Fact

The class of languages Q-REC is closed under $\cup$ and $\cap$.

Let $L_1, L_2 \subseteq \mathcal{T}(A)$ be languages quasi-recognized respectively by $\varphi_1 : \mathcal{T}(A) \to S_1$ and $\varphi_2 : \mathcal{T}(A) \to S_2$, both are quasi-recognized by

$$\begin{aligned} \langle \varphi_1, \varphi_2 \rangle : \mathcal{T}(A) &\to S_1 \times S_2 \\ u &\to (\varphi_1(u), \varphi_2(u)). \end{aligned}$$

Question

Is the class of languages Q-REC closed under product ?

Counterexemple

$\{(1, a^{2n}, 1) \mid n \in \mathbb{N}\} \cdot \{(1, a^{2n}, 1)^{-1} \mid n \in \mathbb{N}\} \notin Q\text{-}REC$

However, the answer is **yes** for languages of **positive** tiles.

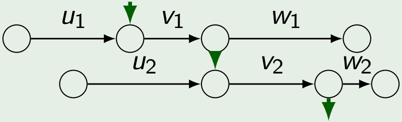
The Monoid of Positive Tiles

Definition

A tile is **positive** when its input occurs before its output.

Positive non-zero tiles can therefore be seen as elements of $\mathcal{T}^+(A) = A^* \times A^* \times A^* \cup \{0\}$, with the product being simpler.

Example (the product $(u_1, v_1, w_1)(u_2, v_2, w_2)$)



It is the concatenation of their roots with matching conditions:

- u_2 is a suffix of $u_1 v_1$ or $u_1 v_1$ is a suffix of u_2,
- w_1 is a prefix of $v_2 w_2$ or $v_2 w_2$ is a prefix of w_1.

The restricted product

Definition (Restricted product)

$u \bullet v$ is defined when $u^L = v^R$, and in this case $u \bullet v = uv$.

Lemma (Preservation of the restricted product)

For any adequate premorphism φ, we have $\varphi(u \bullet v) = \varphi(u) \bullet \varphi(v)$.

We will show closure under **restricted product**, then express the product from the restricted product:

Fact

If the restricted product preserves quasi-recognizability over languages of positive tiles, then the product does too.

$$L_1 L_2 = \Big((A^*)^L L_1 (A^*)^R \bullet L_2\Big) \cup \Big(L_1 \bullet (A^*)^L L_2 (A^*)^R\Big) \cup \Big((A^*)^L L_1 \bullet L_2 (A^*)^R\Big) \cup \Big(L_1 (A^*)^R \bullet (A^*)^L L_2\Big)$$

And Q-REC is closed by product with $(A^*)^L$ or $(A^*)^R$ and by $\cup$.

The Monoid of Restricted Decompositions

For any E-ordered monoid S, we define the set $\mathcal{D}^r(S)$ by

$$\mathcal{D}^r(S) = \{X \in \mathcal{P}(S \times S) \mid \exists c \in S, \\ (c, c^L) \in X, (c^R, c) \in X, \forall (x, y) \in X, x \bullet y = c\}$$

We define the product $*$ from $S \times S$ to $\mathcal{P}(S \times S)$ by

$$(x, x') * (y, y') = \{(x(x'yy')^R, x^L x'yy'), (xx'yy'^R, (xx'y)^L y')\}.$$

We extend $*$ to $\mathcal{D}^r(S)$ in a point-wise manner

$$X * Y = \bigcup_{\substack{(x,x') \in X \\ (y,y') \in Y}} (x, x') * (y, y').$$

$\mathcal{D}^r(S)$ is an E-ordered monoid preordered by $\preccurlyeq$ defined by $X \preccurlyeq Y$ iff for any $x \in X$, there exists $y \in Y$ so that $x \preccurlyeq y$.

Premorphism ψ

Let $\varphi : \mathcal{T}^+(A) \to S$ be an adequate premorphism, we define

$$\begin{aligned}\psi : \mathcal{T}^+(A) &\to \mathcal{D}^r(S)\\ u &\to \{(\varphi(u_1), \varphi(u_2)) \in S \times S \mid u = u_1 \bullet u_2\}\end{aligned}$$

ψ is an adequate premorphism.

Lemma

For any $L_1 = \varphi^{-1}(\varphi(X_1))$ and $L_2 = \varphi^{-1}(\varphi(X_1))$,
$L_1 \bullet L_2 = \psi^{-1}(\psi(\{X \in \mathcal{D}^r(S) \mid X \cap (X_1 \times X_2) \neq \emptyset\}))$.

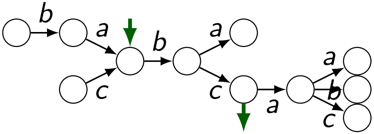
Since for any L_1, L_2 quasi-recognized by respectively φ_1 and φ_2, both are recognized by $\langle \varphi_1, \varphi_2 \rangle$,

Corollary

The product of two quasi-recognizable languages of positive tiles is quasi-recognizable.

To be continued

- Extension to the recognition of Kleene's $*$ over quasi-recognizable languages of positive tiles ?
- Extension to the non-linear case : birooted trees ?



- First-order logic definability ?

Thank you for your attention.