Optimal Sorting Networks

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University of Oxford

Sorting in Hardware

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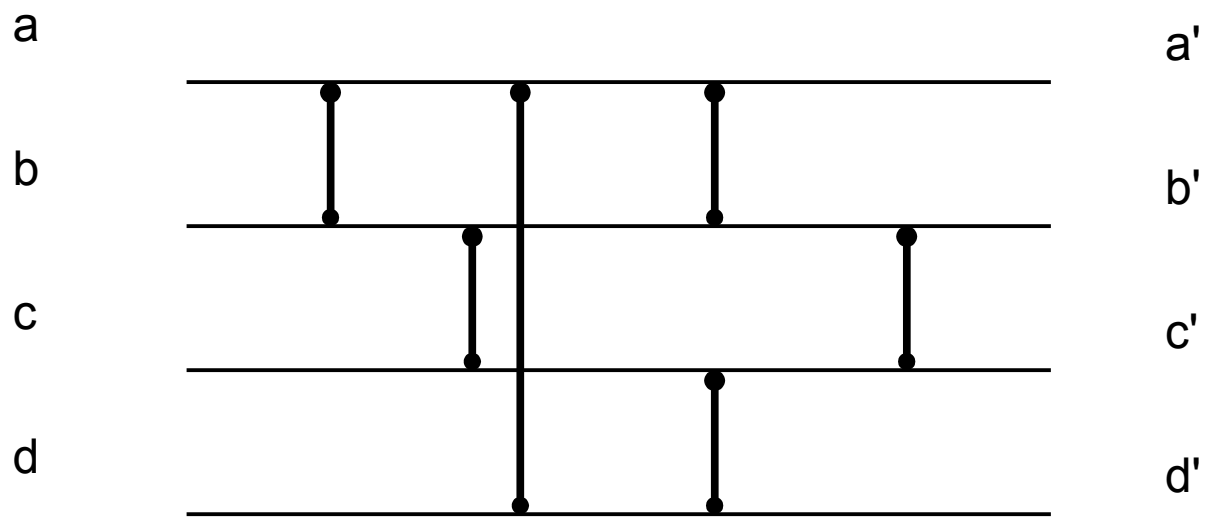

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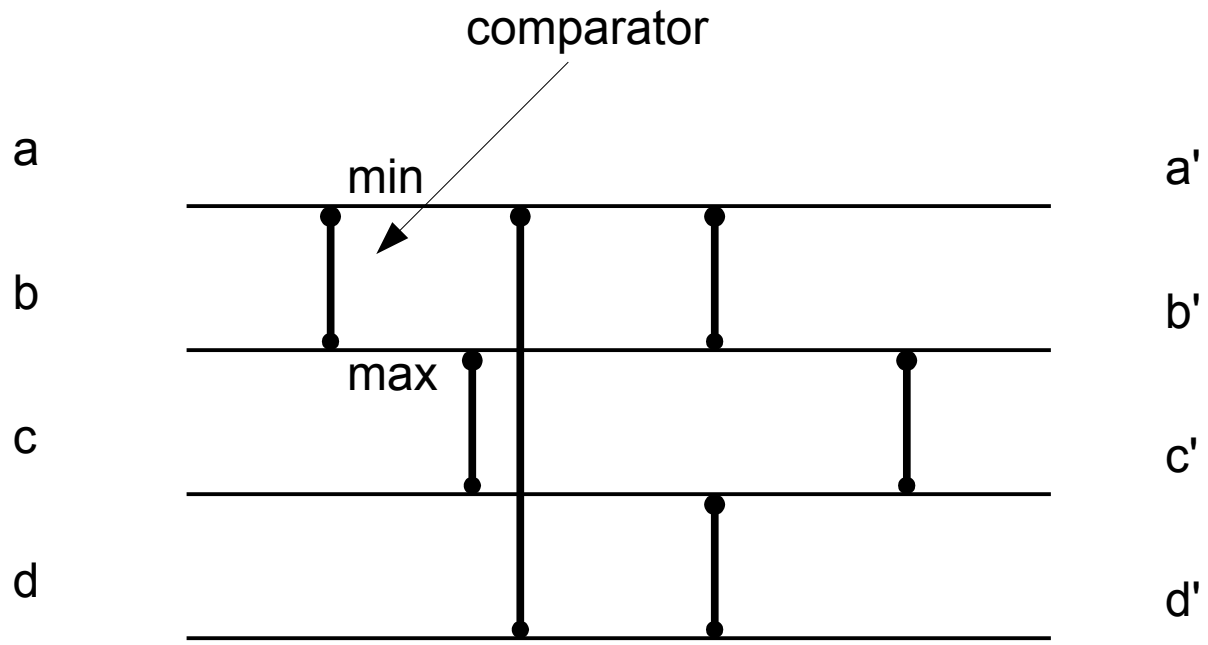

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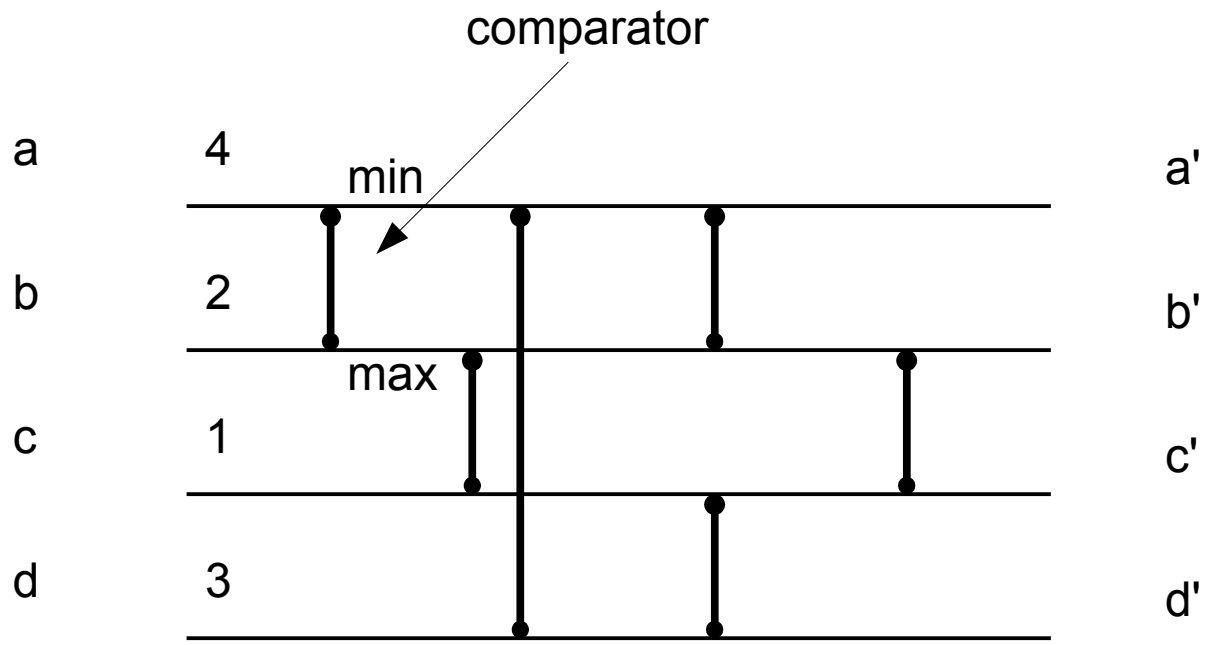

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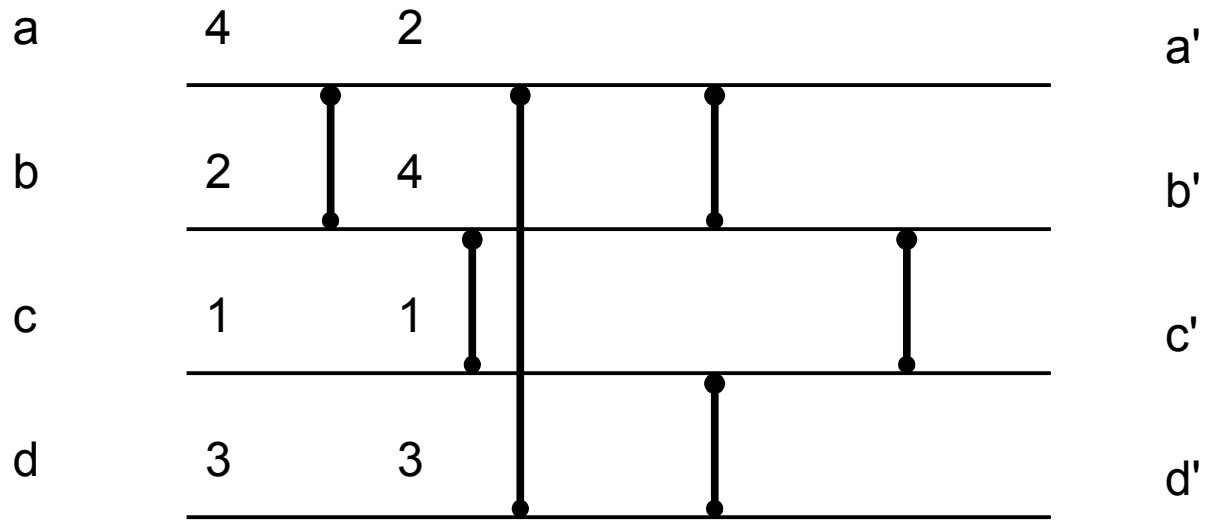

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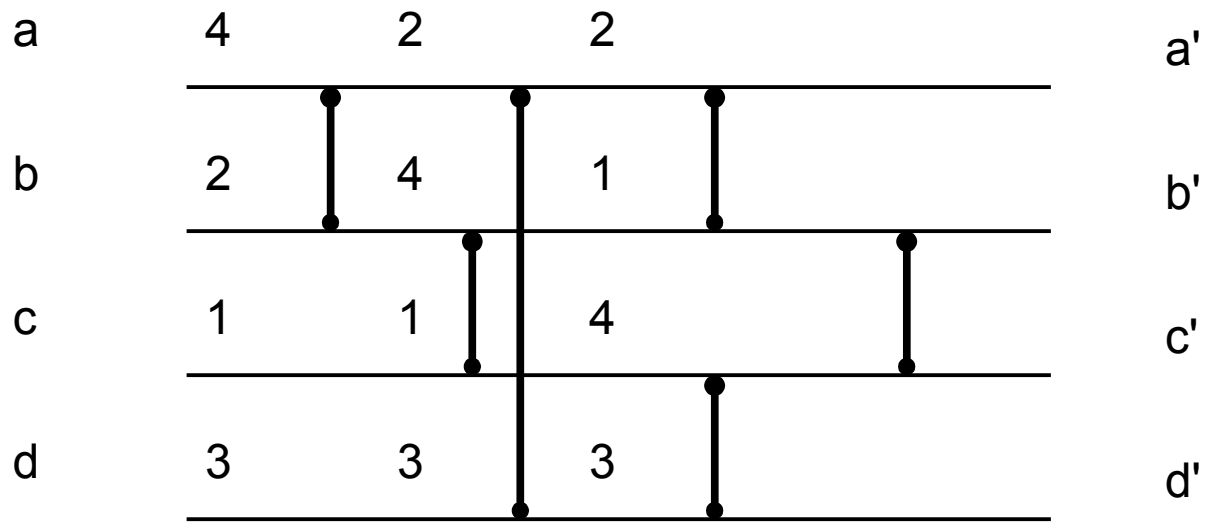

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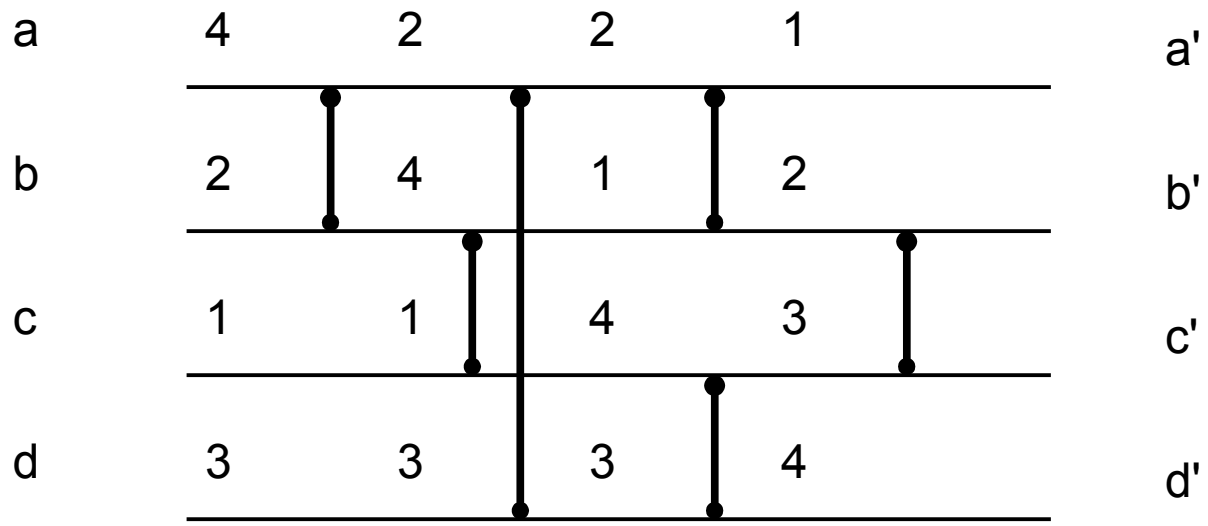

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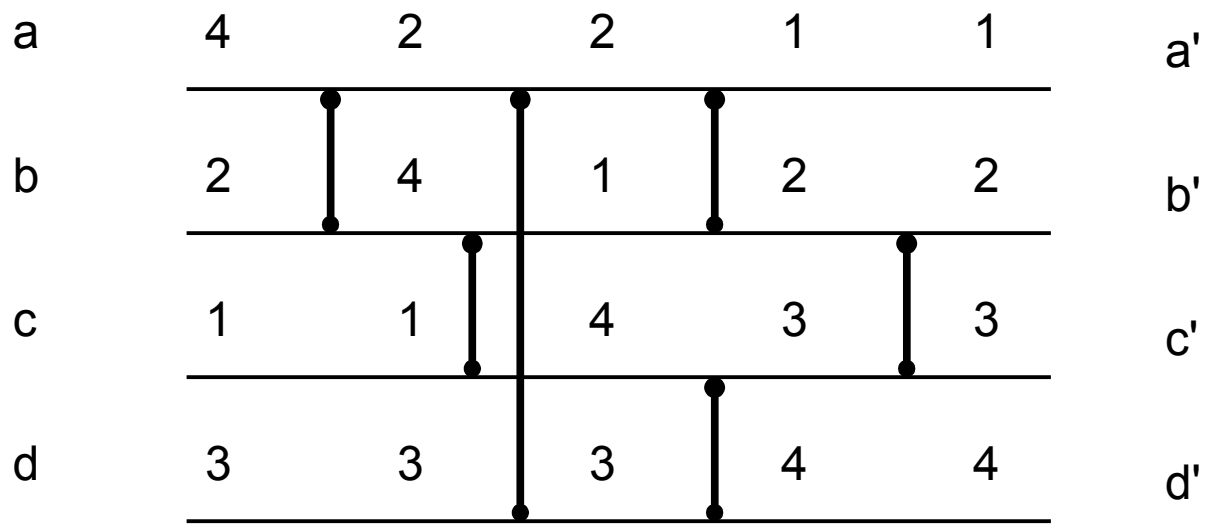

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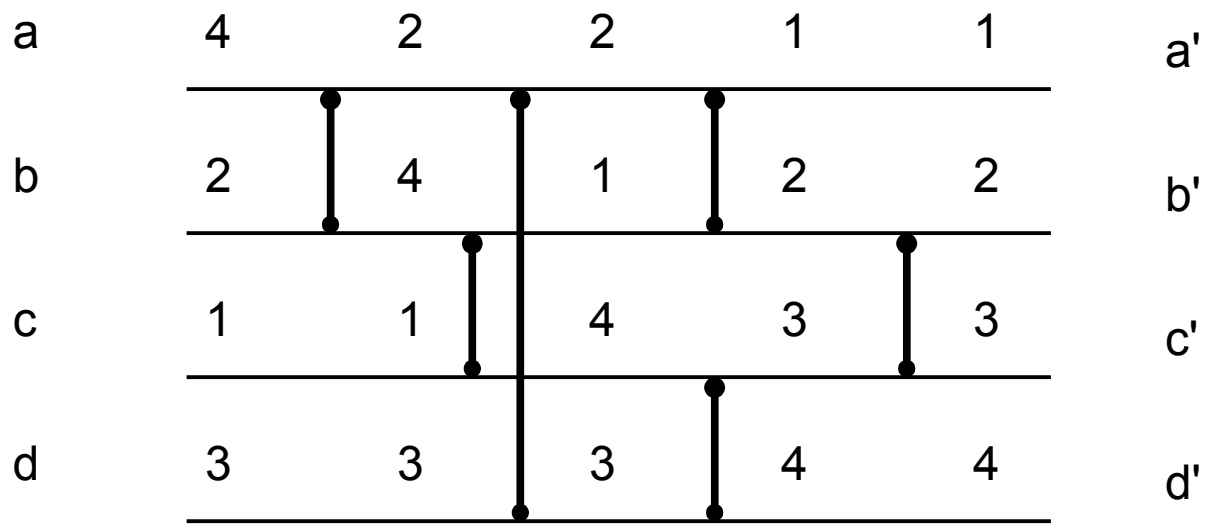

Sorting network – sorts every input

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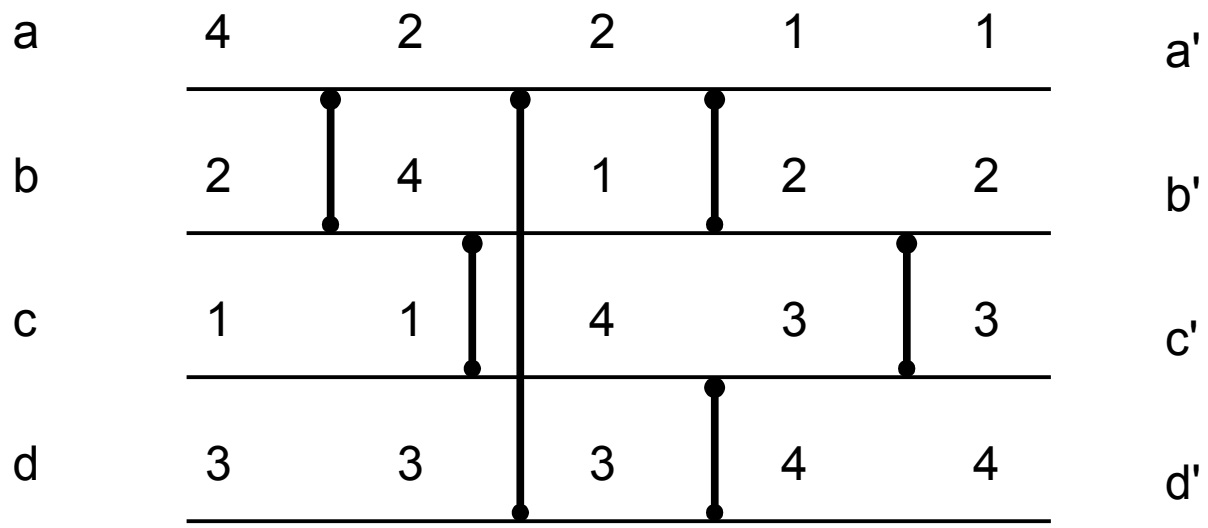

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Time/Depth: 4

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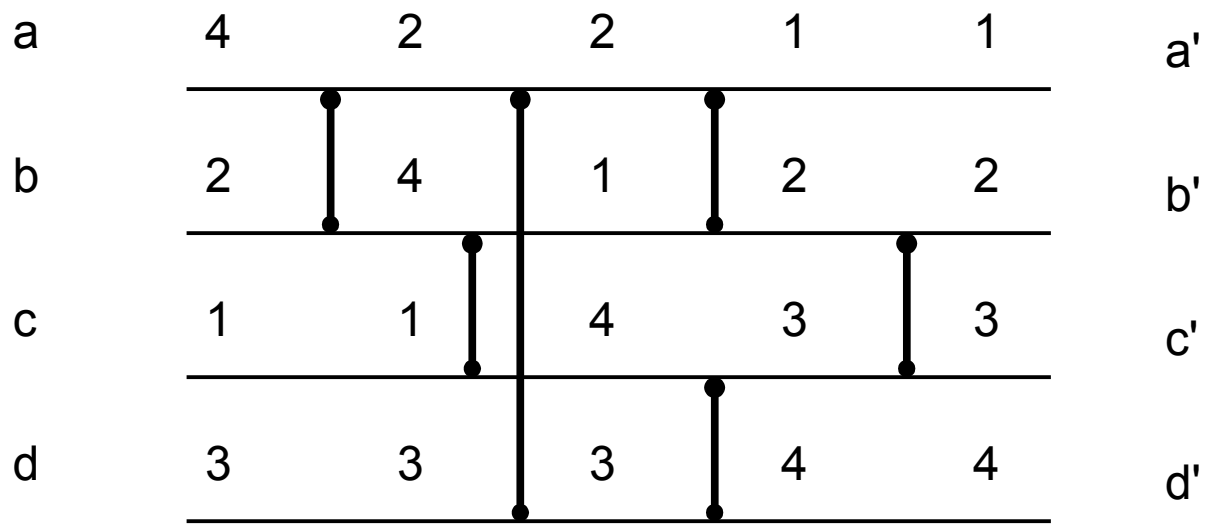

Sorting network – sorts every input

Time/Depth: 4

What is the fastest way to sort n inputs?

General n: Ajtai, Komlos, Szemeredi (1983) – Θ (log n)

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Specific n: Knuth (1973) – The Art of Computer Programming

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Upper Bound	0	1	3	3	5	5	6	6	7	7	8	8	9	9	9	9
Lower Bound	0	1	3	3	5	5	6	6	6	6	6	6	6	6	6	6

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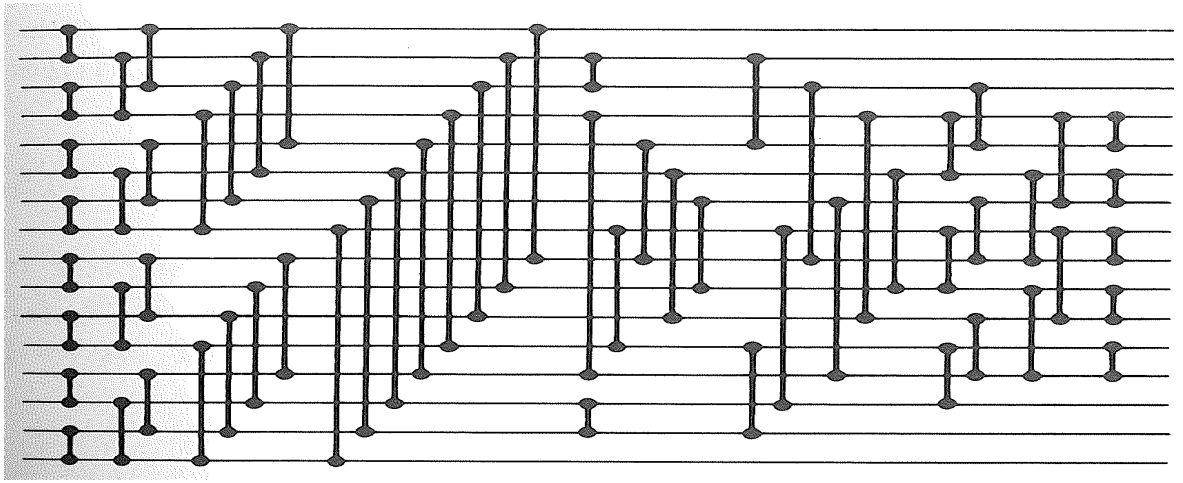
Knuth (1973) – The Art of Computer Programming, Vol. 3



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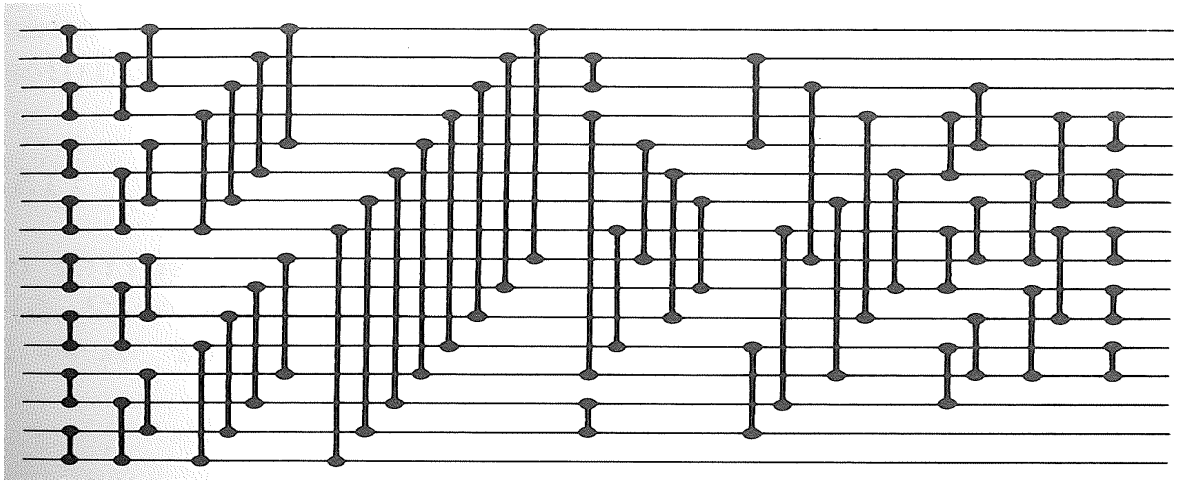

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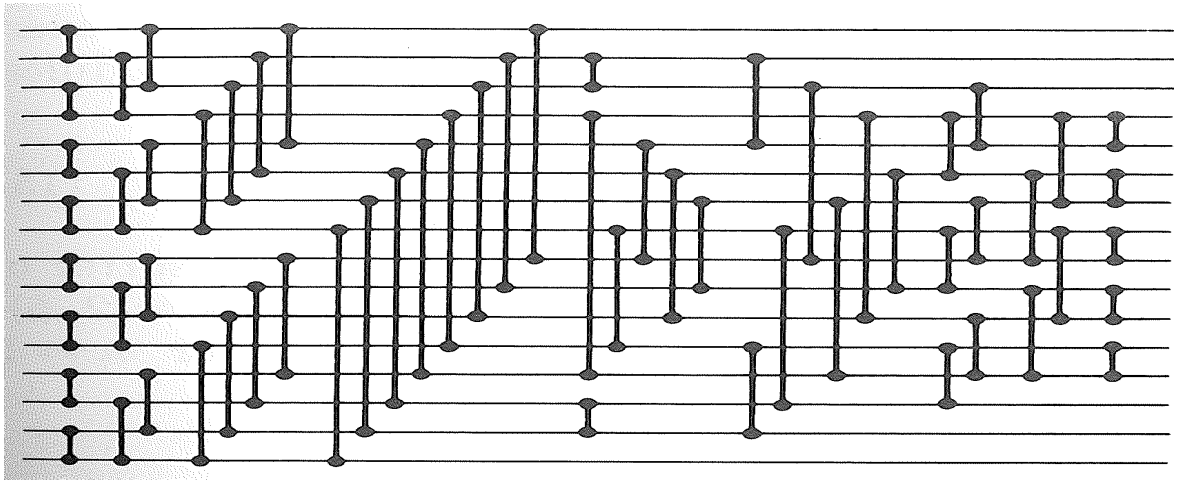

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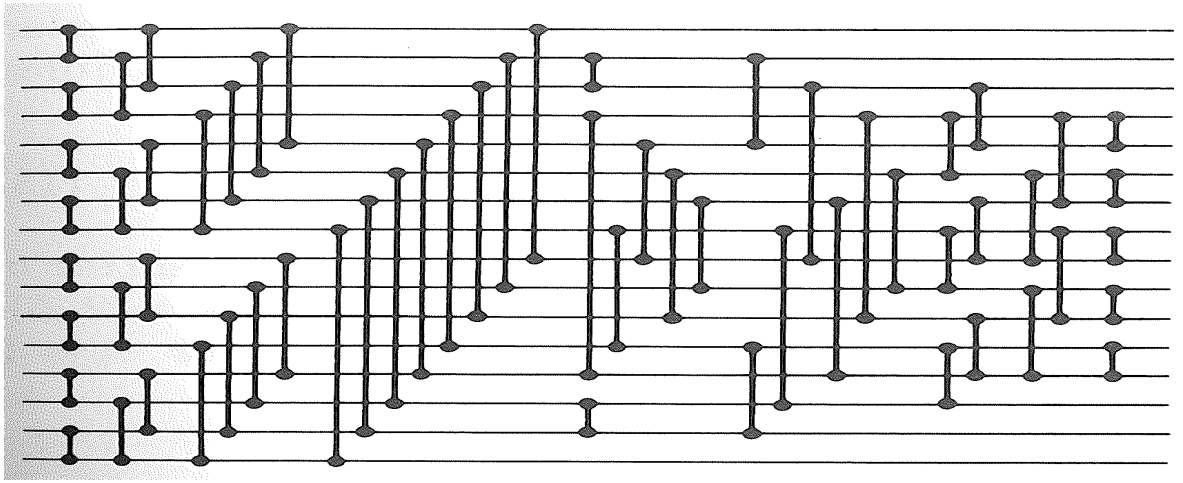

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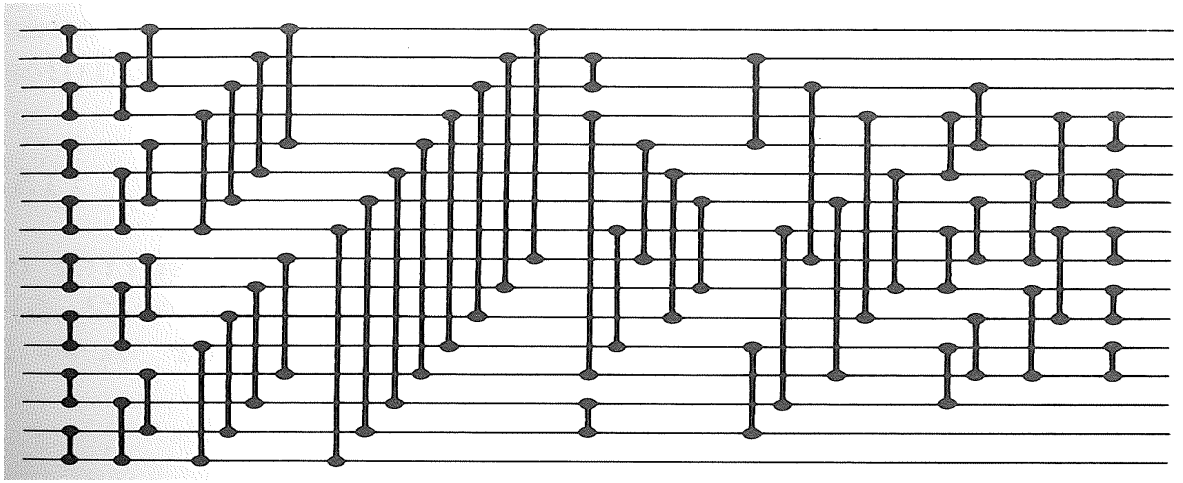

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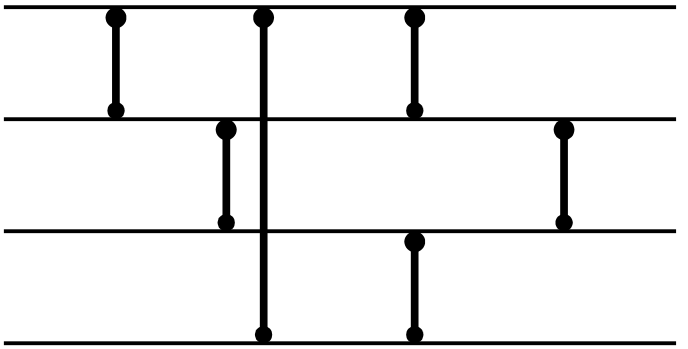
35696 candidates for each layer



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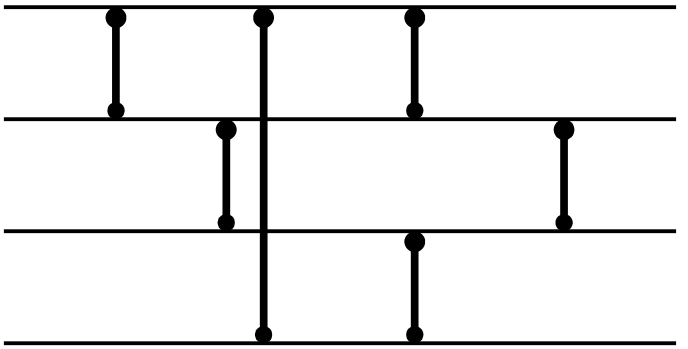
$35696^7 > 7 \times 10^{31}$ networks



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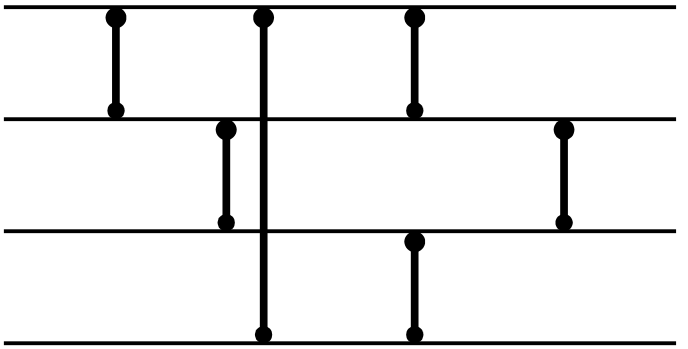

Build propositional formula satisfiable iff a depth 7 network sorts 11 inputs

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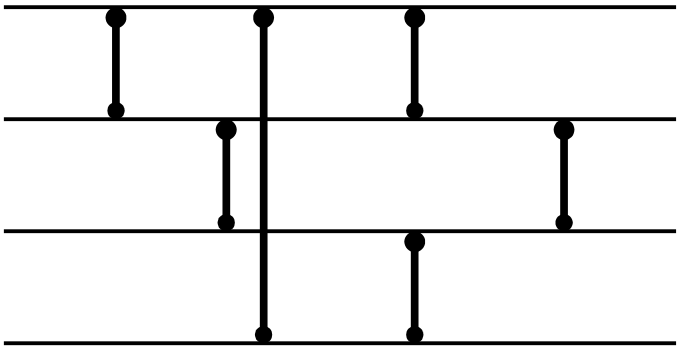

Build propositional formula satisfiable iff a depth 7 network sorts 11 inputs

Variables: C(i,j,l) – true if comparator (i,j) in level l

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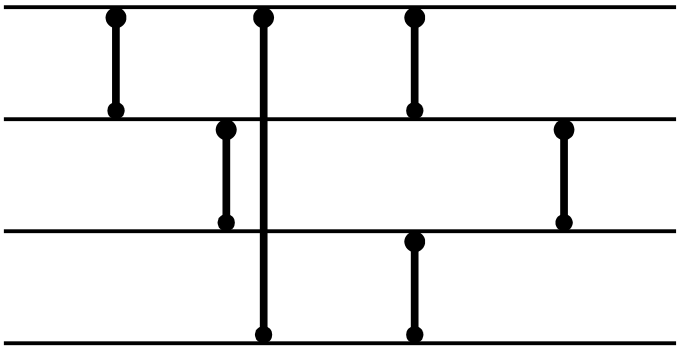

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Formula: $\left(\bigwedge_{\substack{\text{line} \\ 1 \le i \le 11}} \bigwedge_{\substack{\text{layer} \\ 1 \le l \le 7}} i \text{ used at most once at layer } l \right) \bigwedge \text{network sorts every input}$

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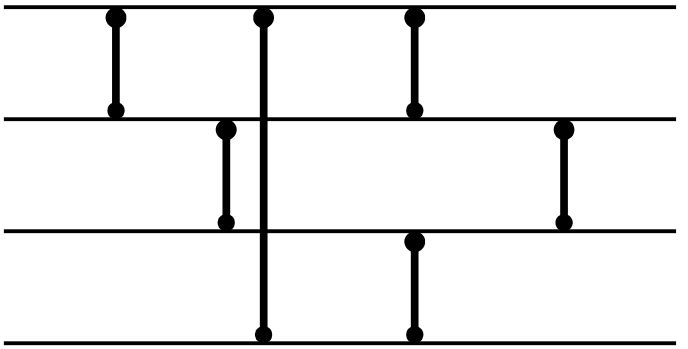

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Lemma: Network sorts every input iff it sorts every binary input

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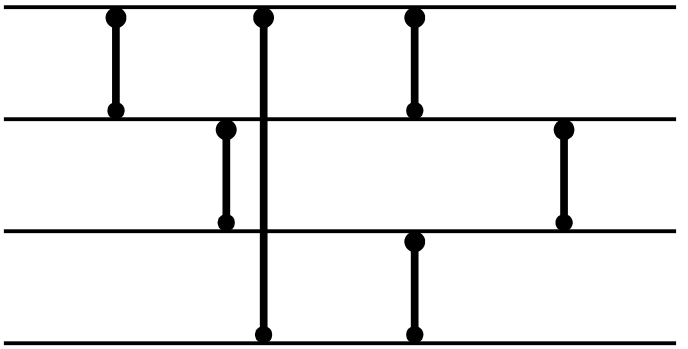

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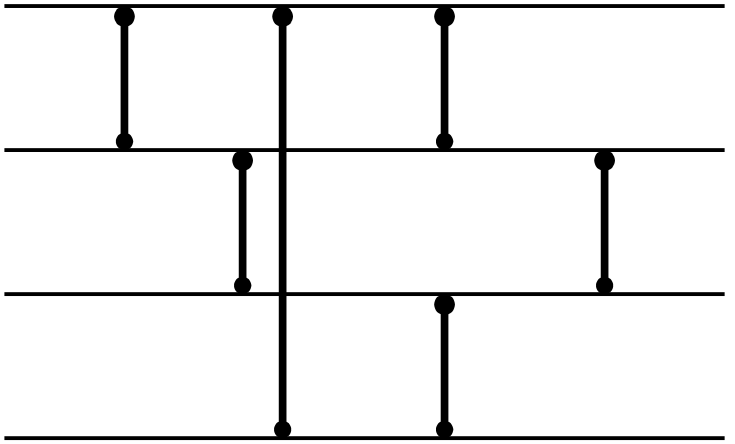
Lemma: Network sorts every input iff it sorts every binary input

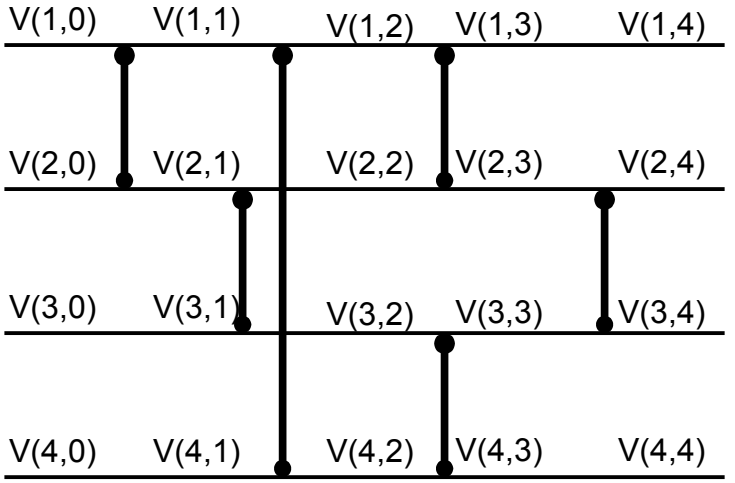
Network sorts every input:
sorts(00000000000) and sorts(00000000001) and sorts(00000000010) and ... and sorts(11111111111)

sorts(01101001011) =



Variables: V(i,l) – value of line i at level l

sorts(01101001011) =



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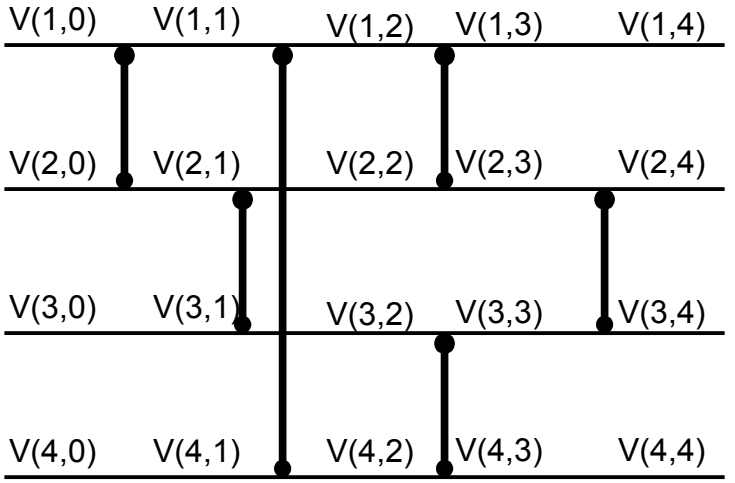
sorts(01101001011) =

V(_,0) = 01101001011 and
V(_,7) = 00000111111 and



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if C(1,2,1) then V(1,1) = min(V(1,0),V(2,0))
V(2,1) = max(V(1,0),V(2,0))
if C(1,3,1) then V(1,1) = min(V(1,0),V(3,0))
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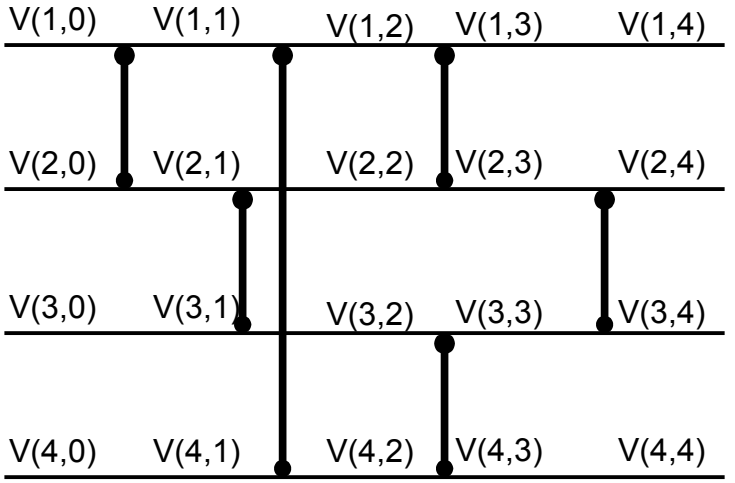

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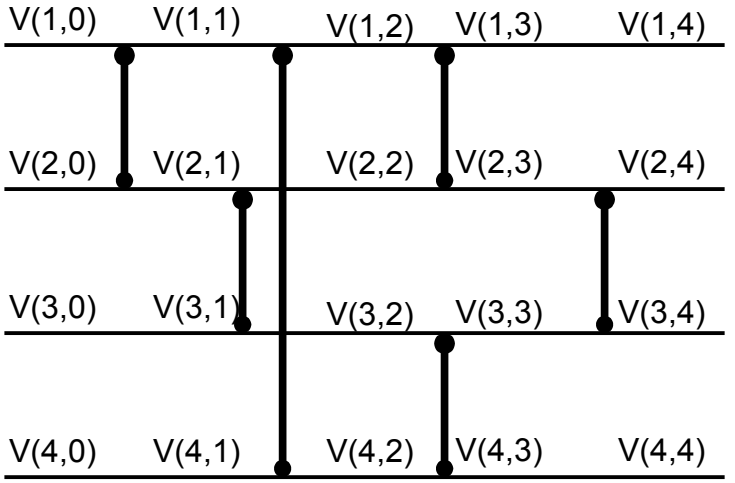

min(x,y) = x and y
max(x,y) = x or y

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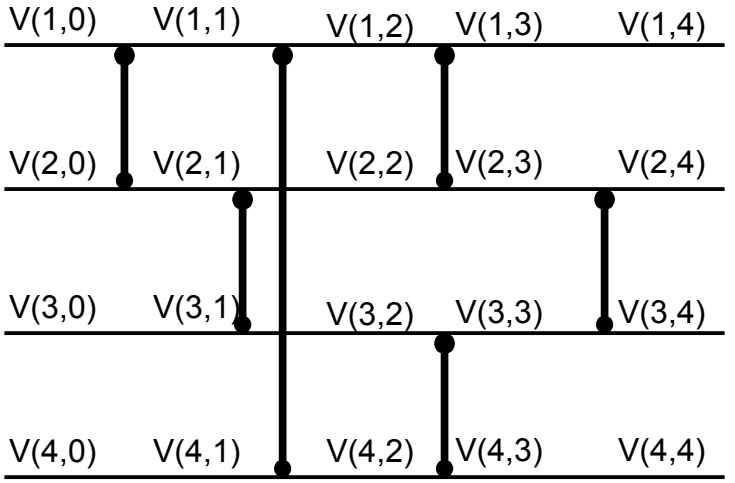

min(x,y) = x and y
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Can build formula F(n,d) which is satisfiable iff some network of depth d sorts n inputs

Variables: V(i,l) – value of line i at level l

sorts(01101001011) =

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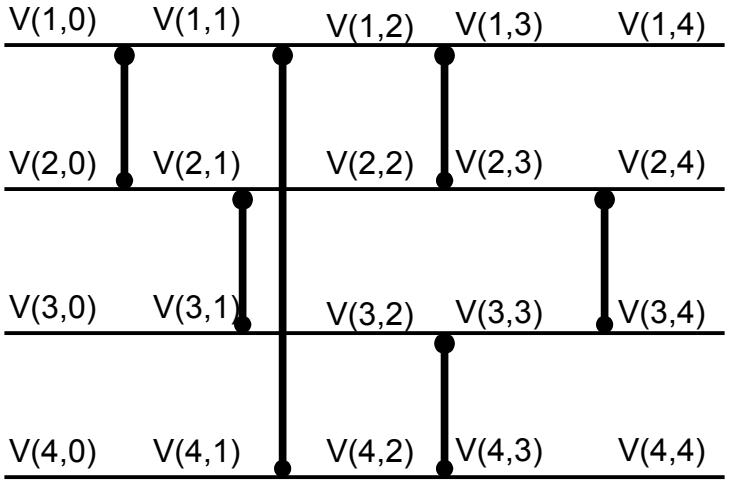

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Can build formula F(n,d) which is satisfiable iff some network of depth d sorts n inputs

n	3	4	5	6	7	8	9	10	11	12	13	14	15	16
d	2	2	4	4	5	5	6	6	7	7	8	8	8	8
UNSAT	<1s	<1s	<1s	<1s	2s	2s	15m	29m	-	-	-	-	-	-
d	3	3	5	5	6	6	7	7	8	8	9	9	9	9
SAT	<1s	<1s	<1s	<1s	<1s	1s	1m	16m	21m	-	-	-	-	-

Variables: V(i,l) – value of line i at level l

sorts(01101001011) =

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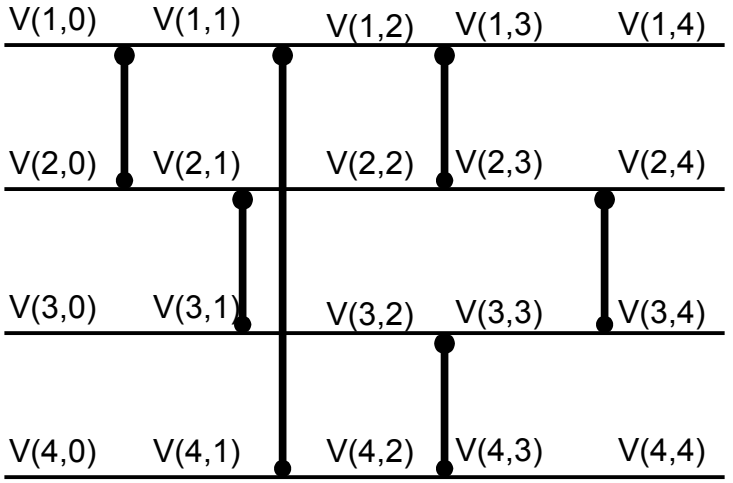

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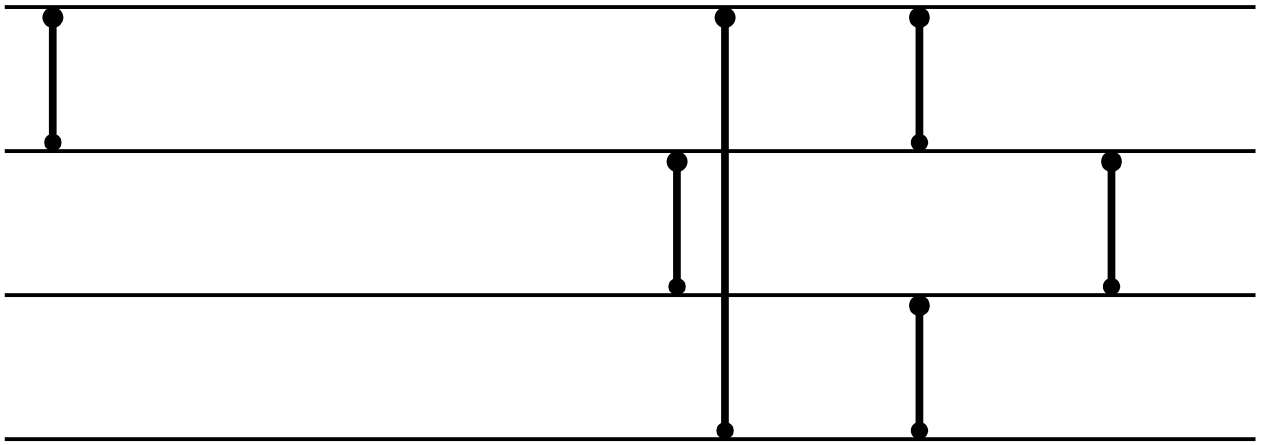
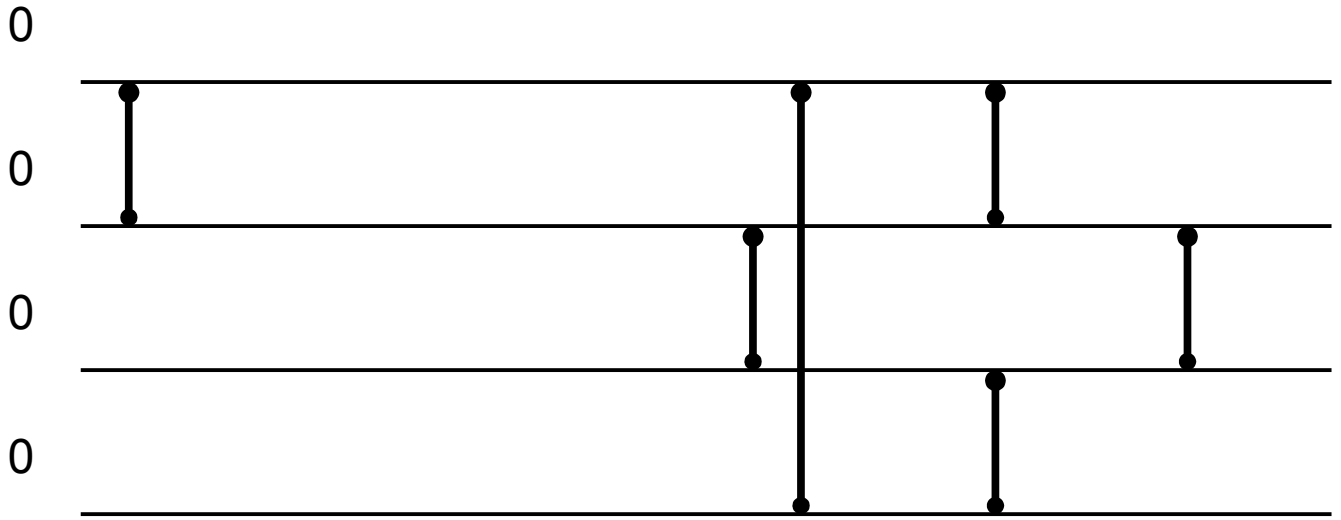
d	2	2	4	4	5	5	6	6	7	7	8	8	8	8
UNSAT	<1s	<1s	<1s	<1s	2s	2s	15m	29m	-	-	-	-	-	-
d	3	3	5	5	6	6	7	7	8	8	9	9	9	9
SAT	<1s	<1s	<1s	<1s	<1s	1s	1m	16m	21m	-	-	-	-	-

How to restrict set of possible networks?

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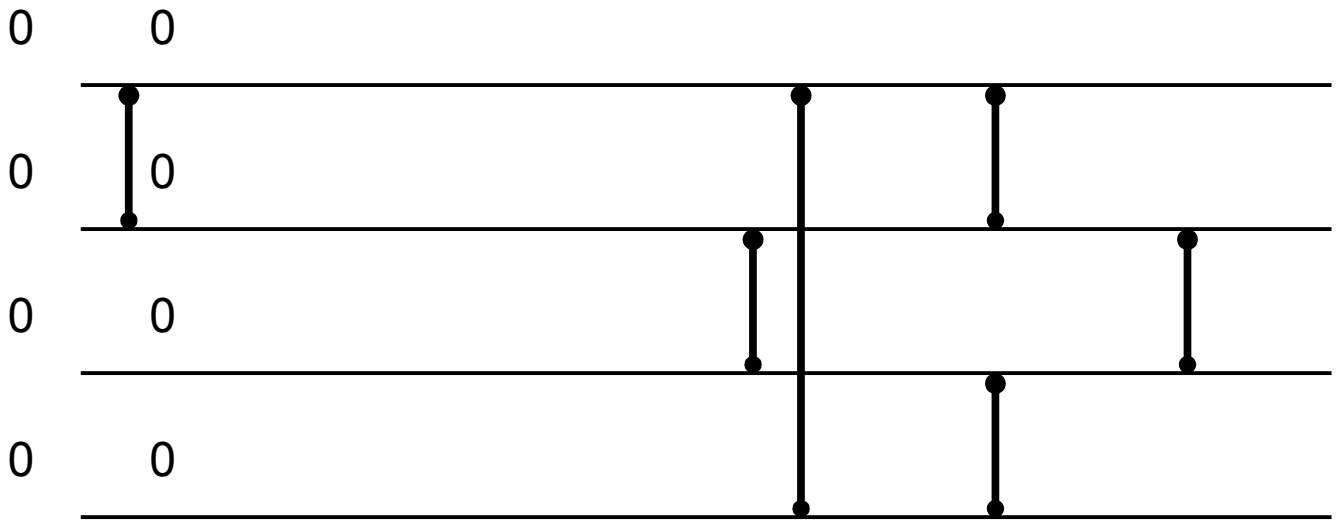
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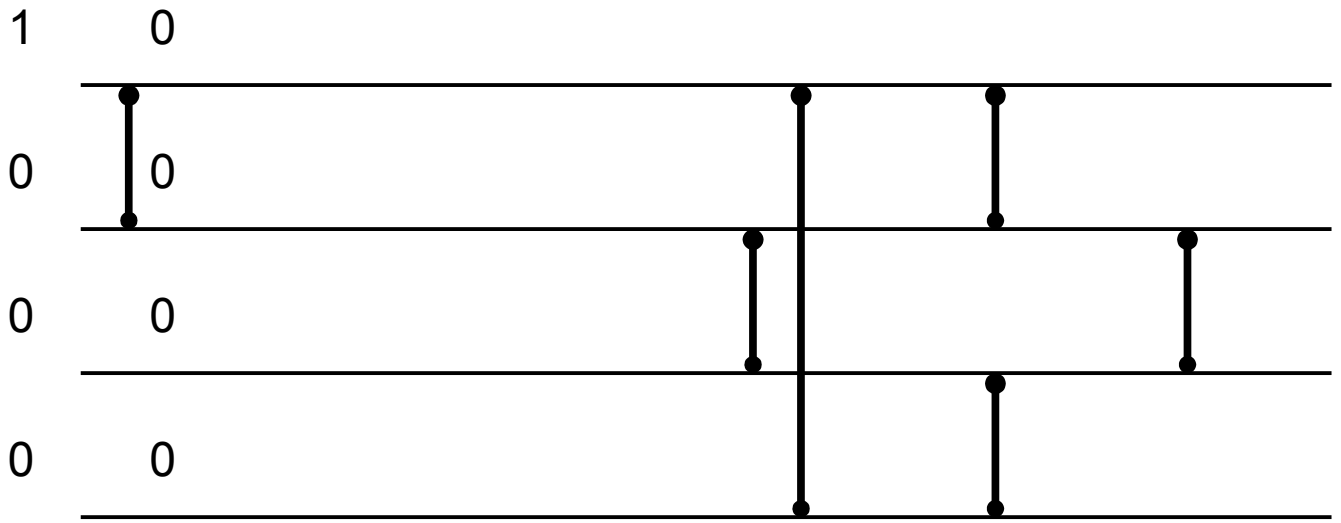
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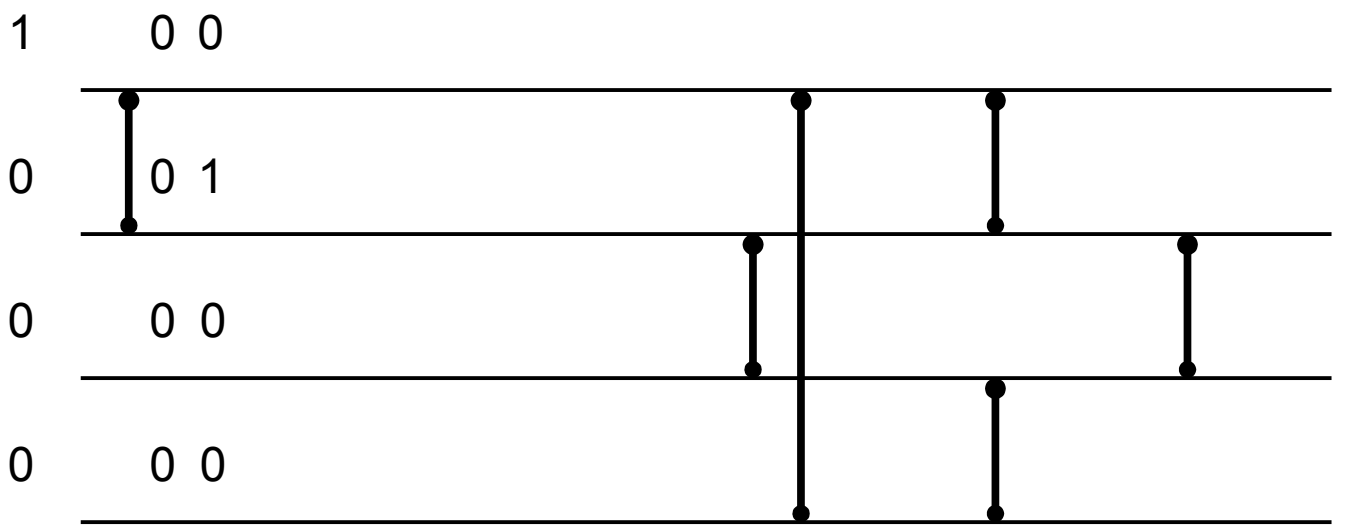
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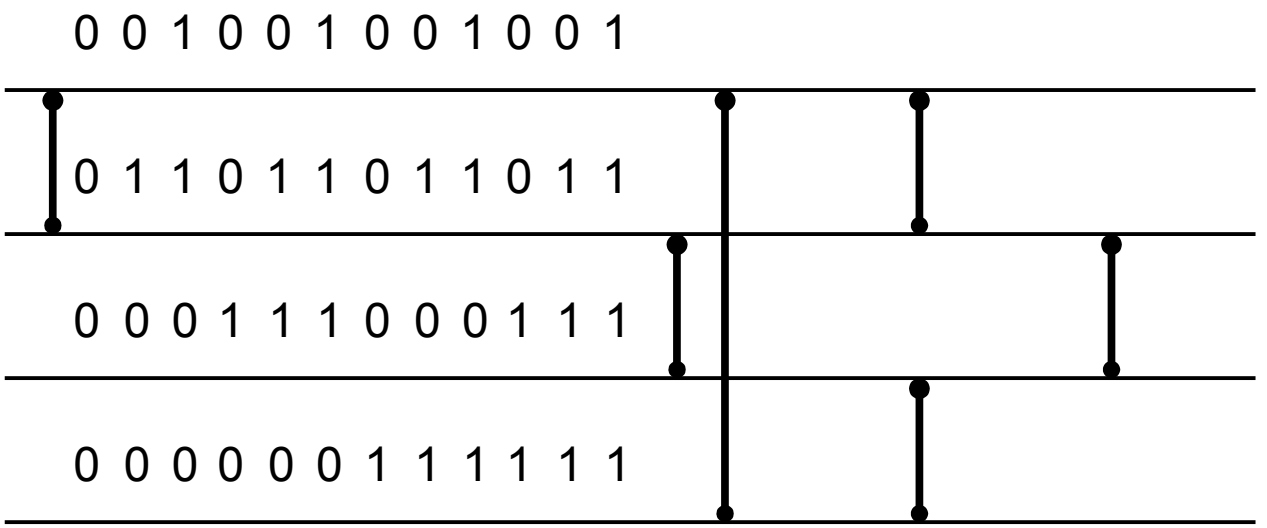
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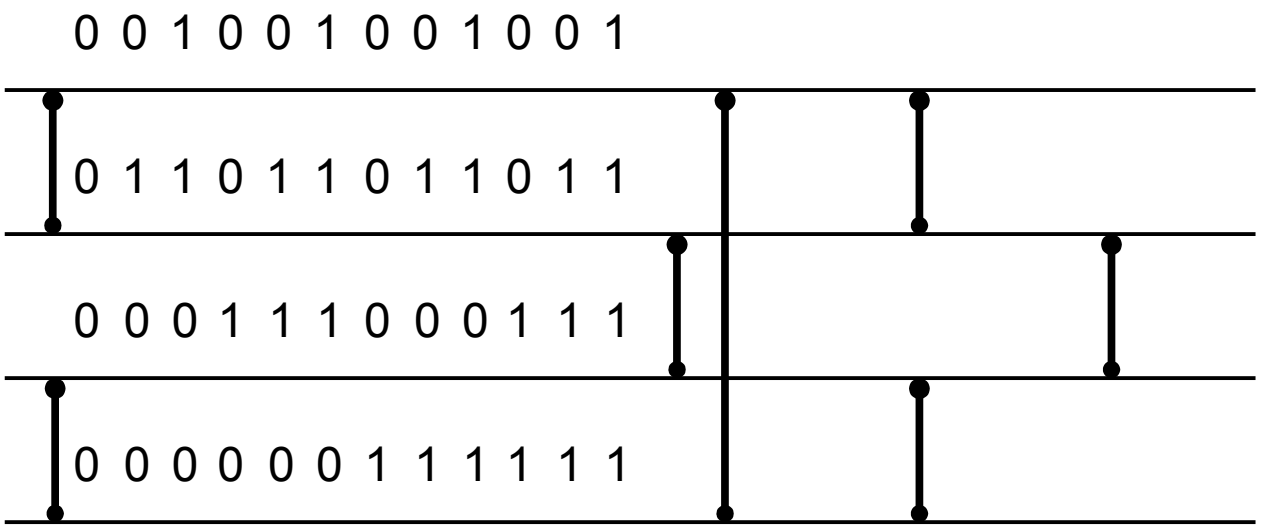
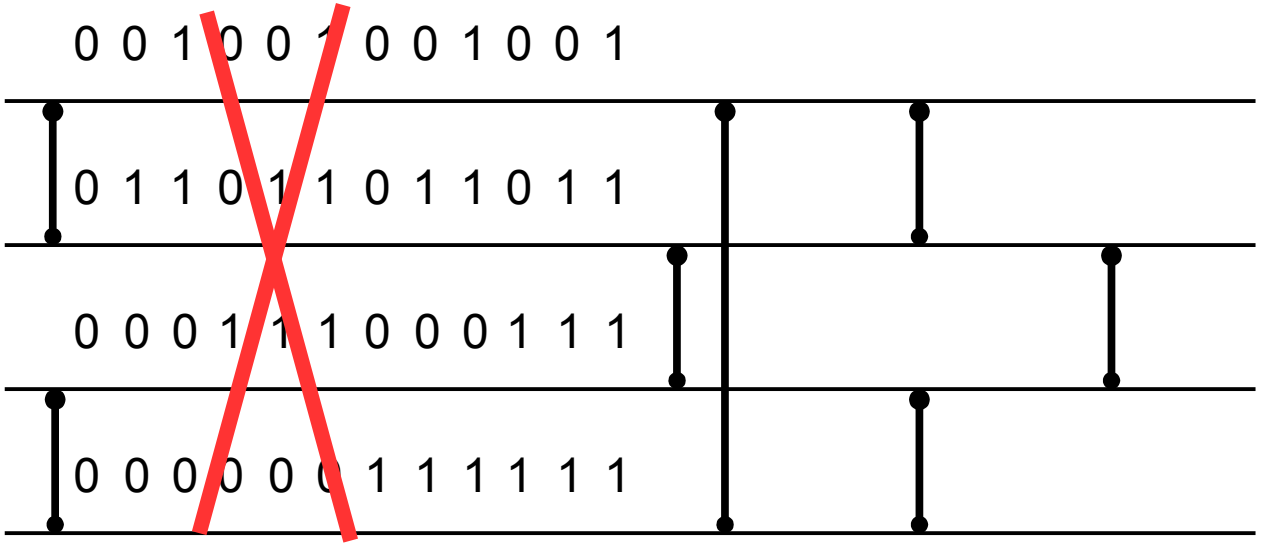
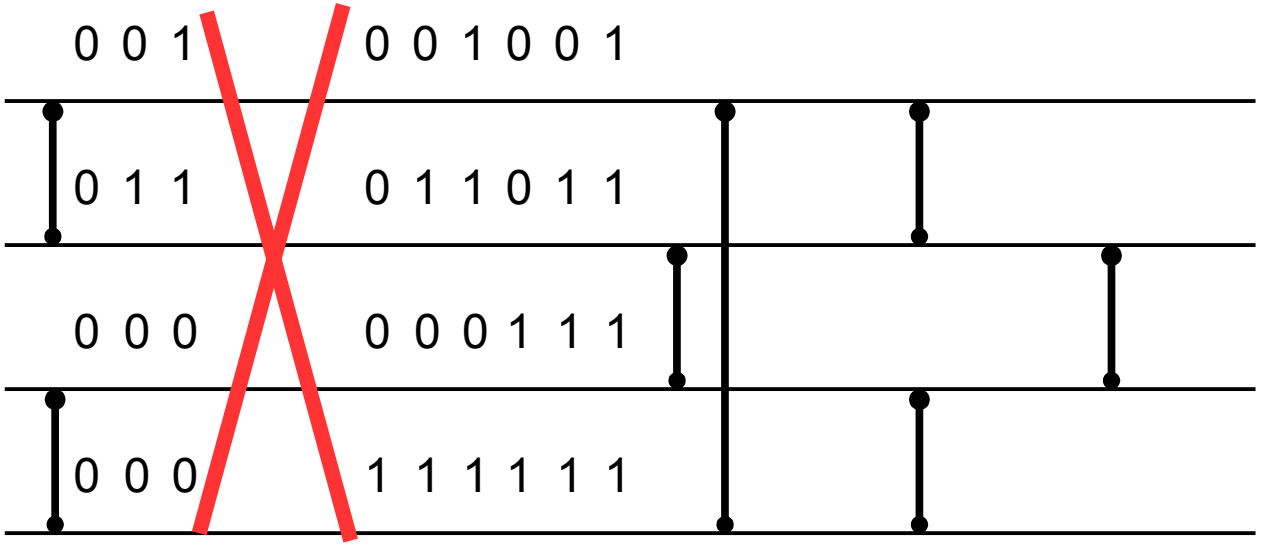
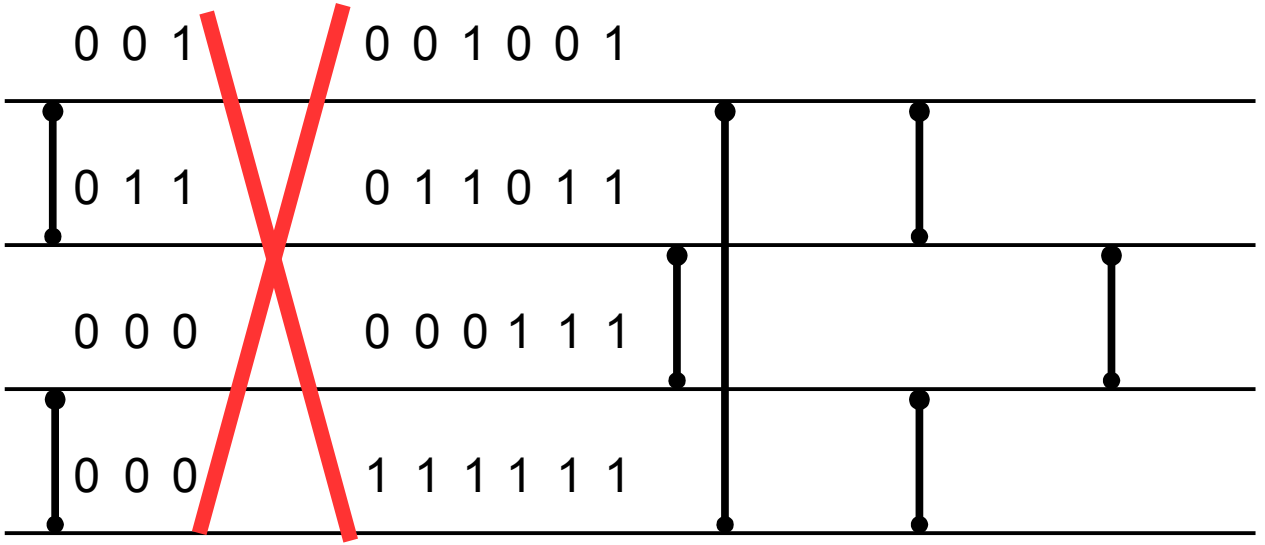
0 1 1 0 1 1 0 1 1 0 1 1
0 0 0 1 1 1 0 0 0 1 1 1
0 0 0 0 0 0 1 1 1 1 1 1

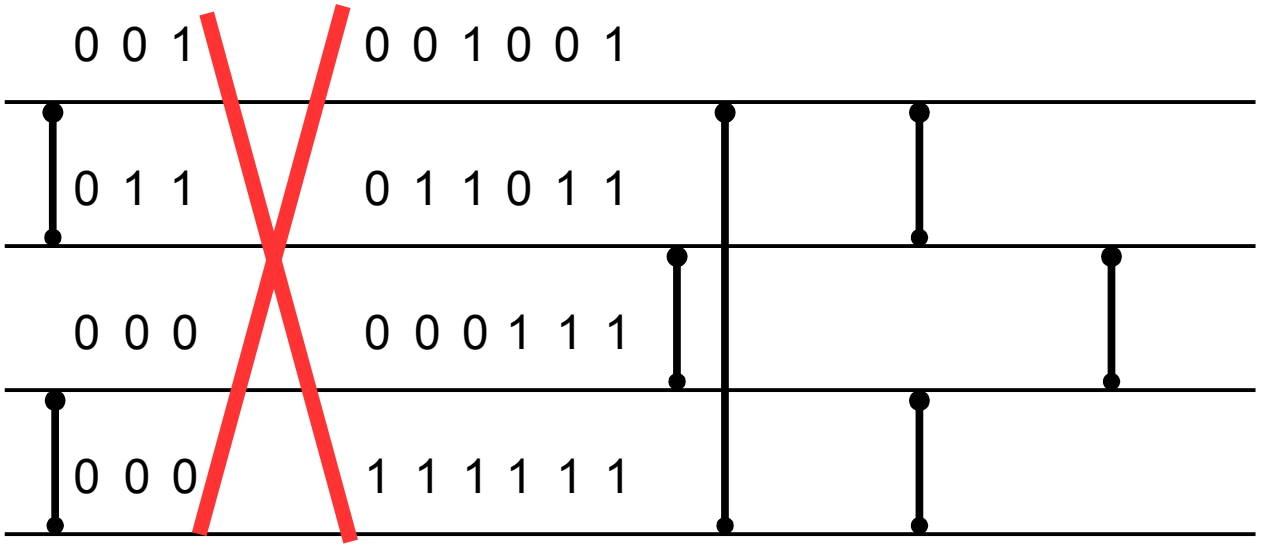

0 0 1 0 0 1 0 0 1 0 0 1
0 1 1 0 1 1 0 1 1 0 1 1
0 0 0 1 1 1 0 0 0 1 1 1
0 0 0 0 0 0 1 1 1 1 1 1


0 0 1 0 0 1 0 0 1 0 0 1
0 1 1 0 1 1 0 1 1 0 1 1
0 0 0 1 1 1 0 0 0 1 1 1
0 0 0 0 0 0 1 1 1 1 1 1

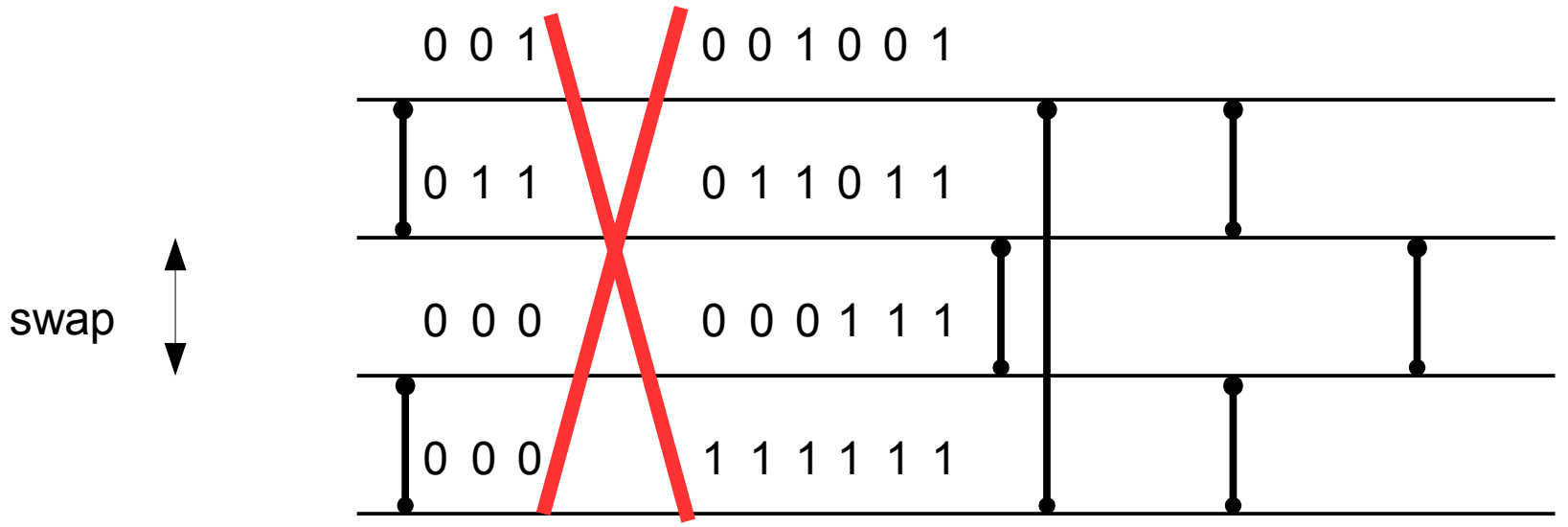

0 0 1
0 0 1 0 0 1
0 1 1
0 1 1 0 1 1
0 0 0
0 0 0 1 1 1
0 0 0
1 1 1 1 1 1



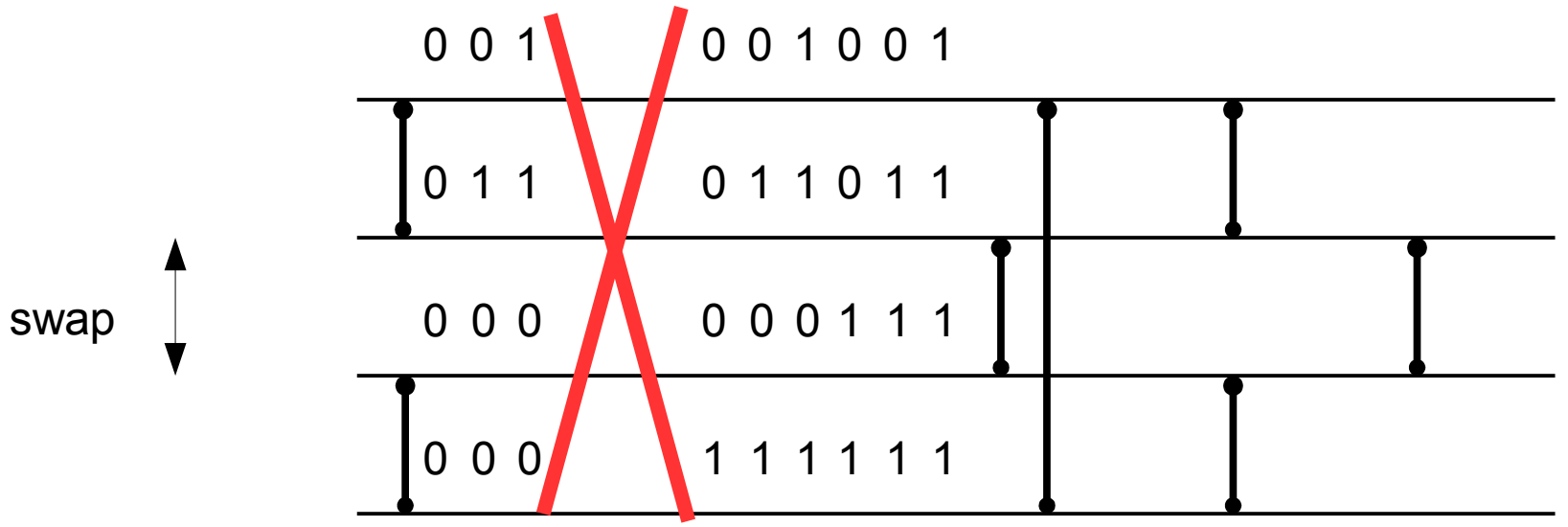
First layer is maximal



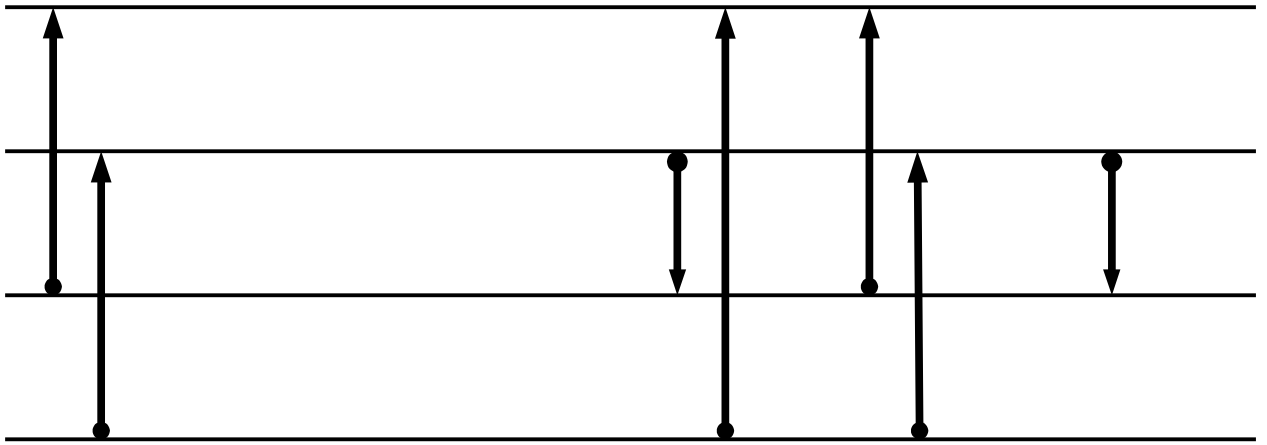
First layer is maximal

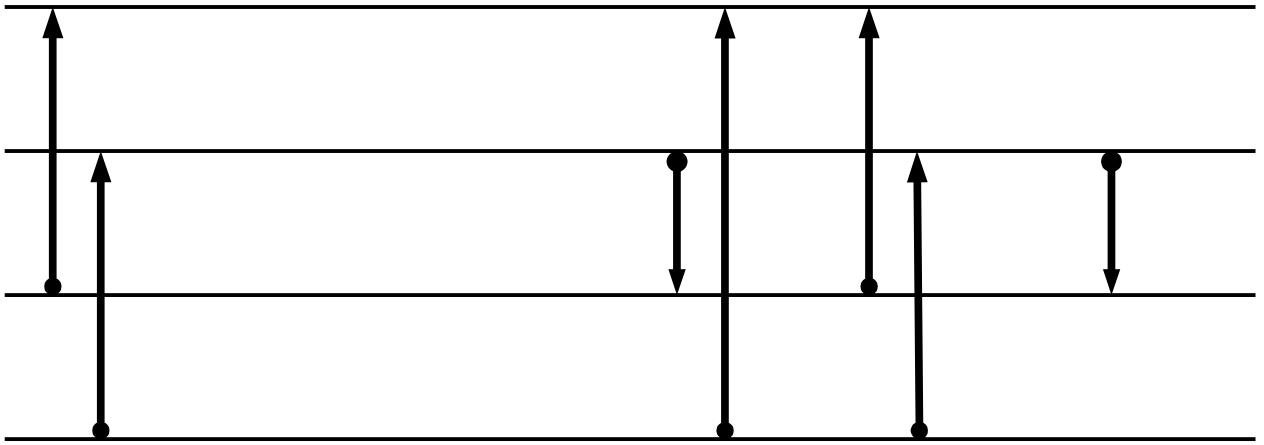


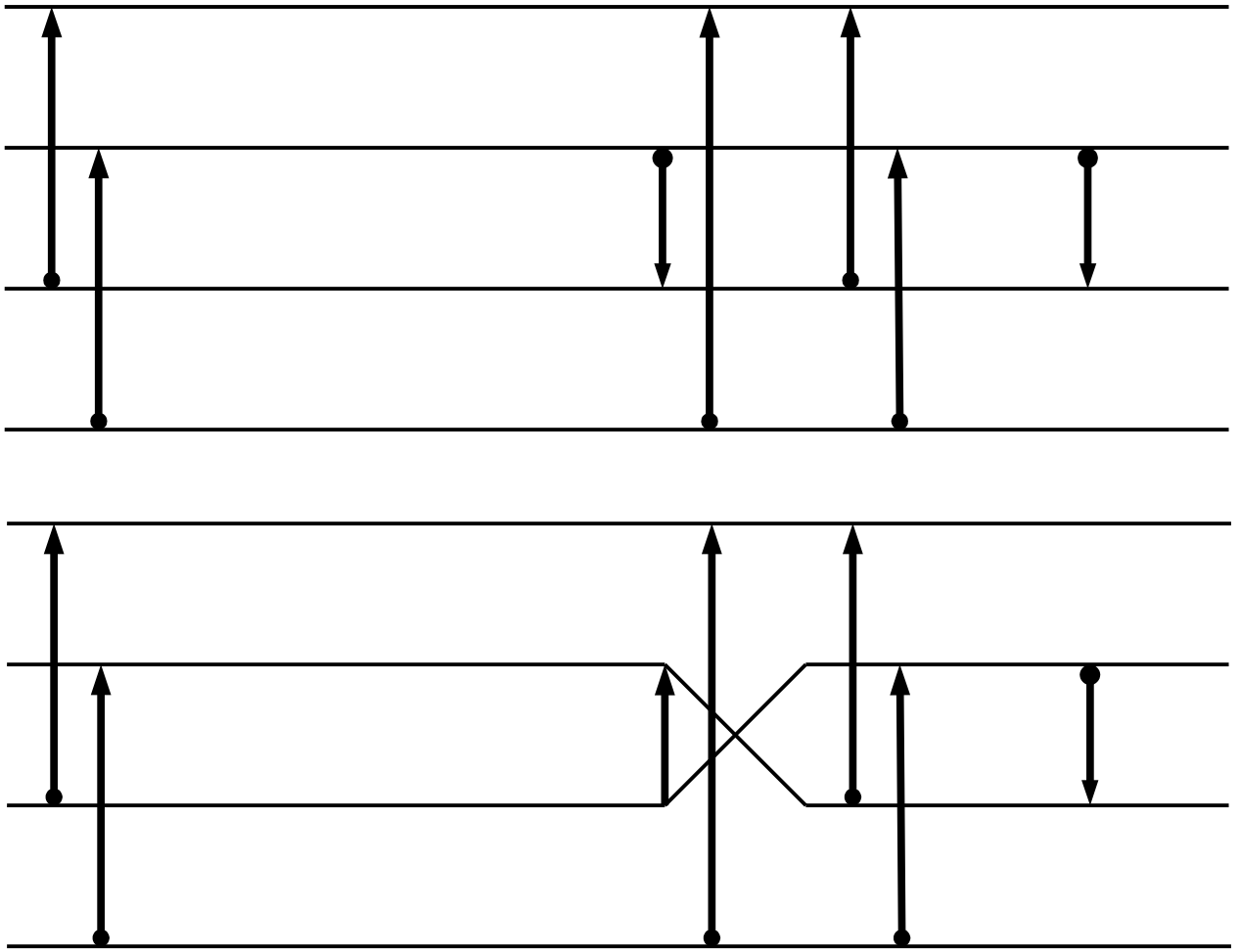
First layer is maximal

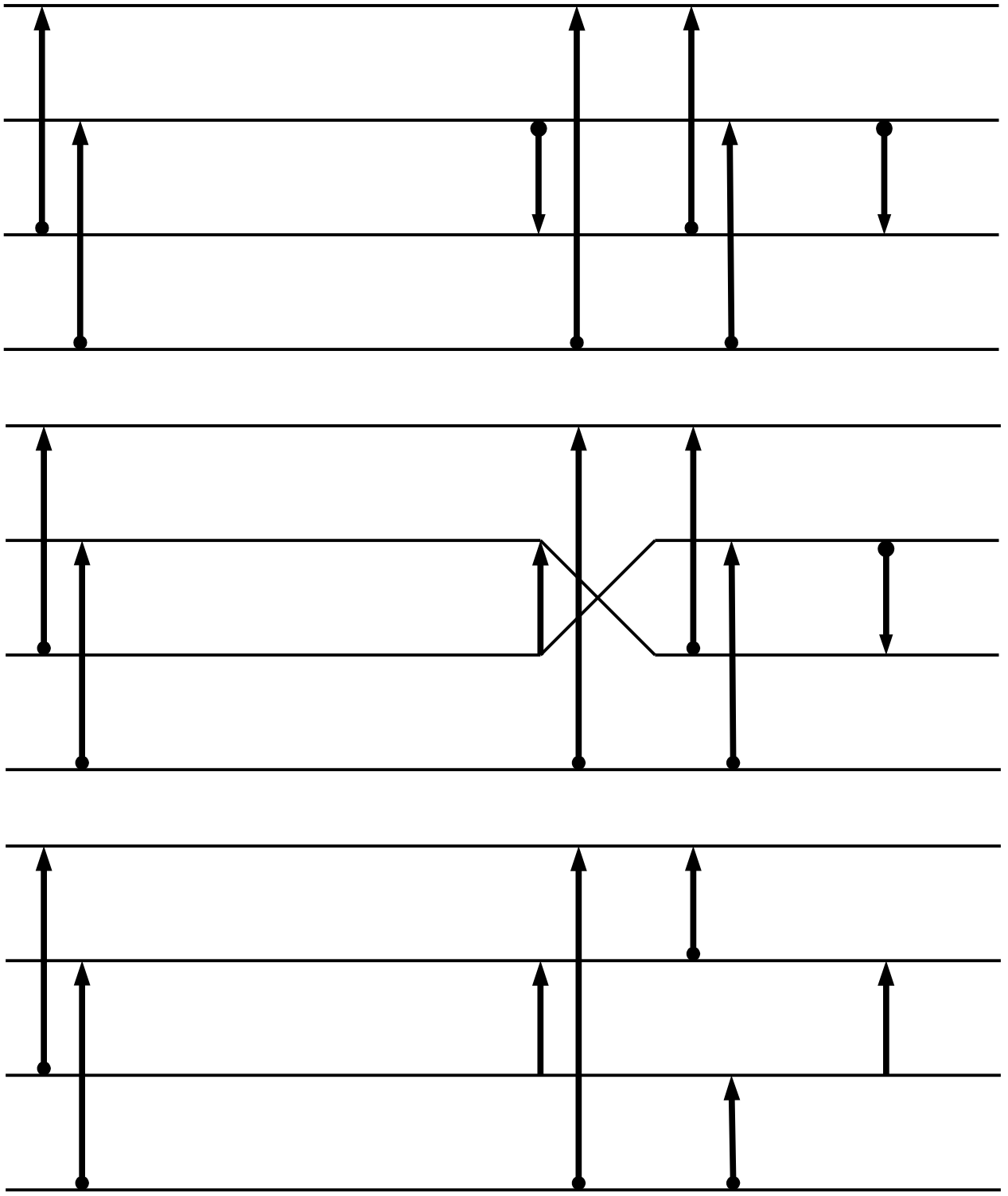


First layer is maximal

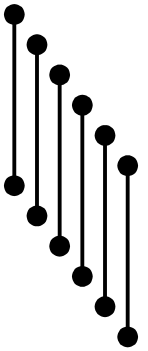


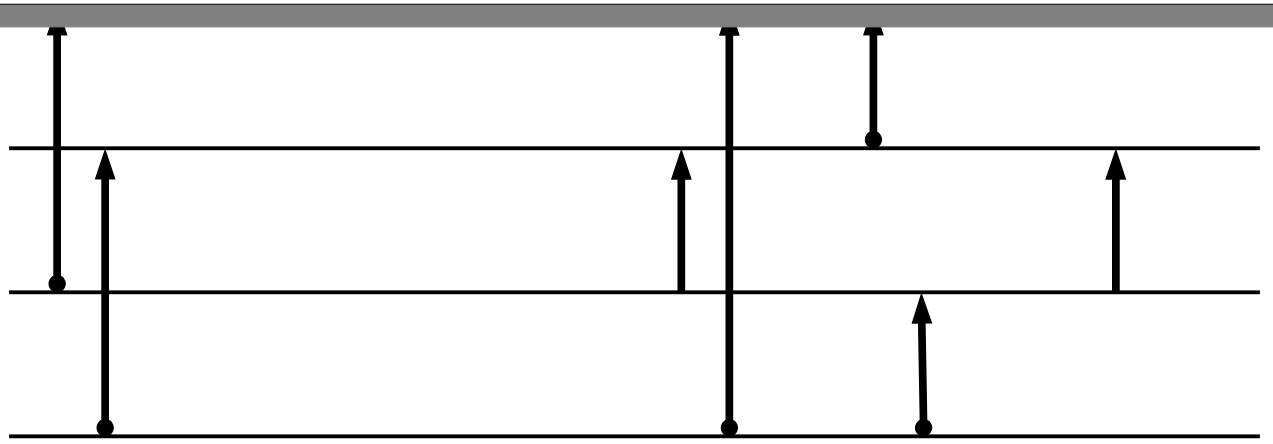




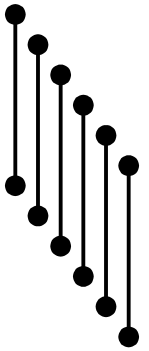


First layer is fixed:



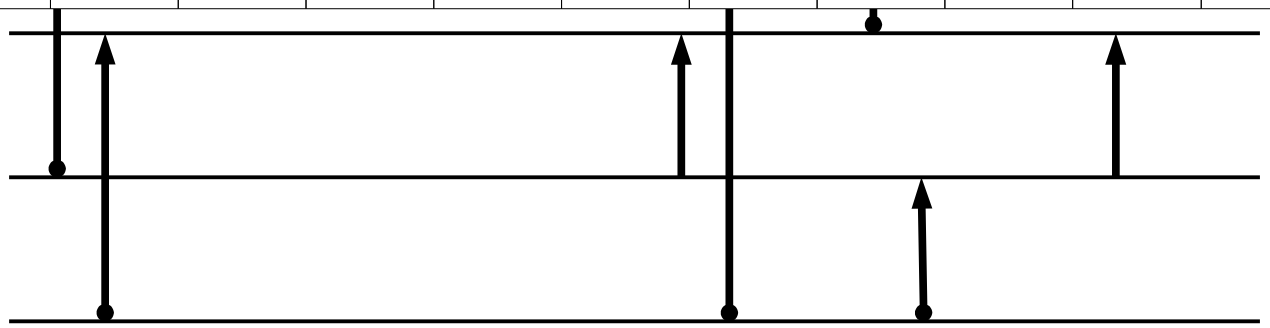


First layer is fixed:

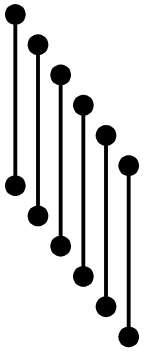


Can set C(i,j,1) variables for the first layer

n	3	4	5	6	7	8	9	10	11	12	13	14	15	16
d	2	2	4	4	5	5	6	6	7	7	8	8	8	8
UNSAT	<1s	<1s	<1s	<1s	2s	2s	15m	29m	-	-	-	-	-	-
First Fixed	<1s	<1s	<1s	<1s	<1s	<1s	2s	10s	13m	23m	-	-	-	-
d	3	3	5	5	6	6	7	7	8	8	9	9	9	9
SAT	<1s	<1s	<1s	<1s	<1s	1s	1m	16m	21m	-	-	-	-	-
First Fixed	<1s	<1s	<1s	<1s	<1s	<1s	1s	19s	4m	12m	-	-	-	-



First layer is fixed:



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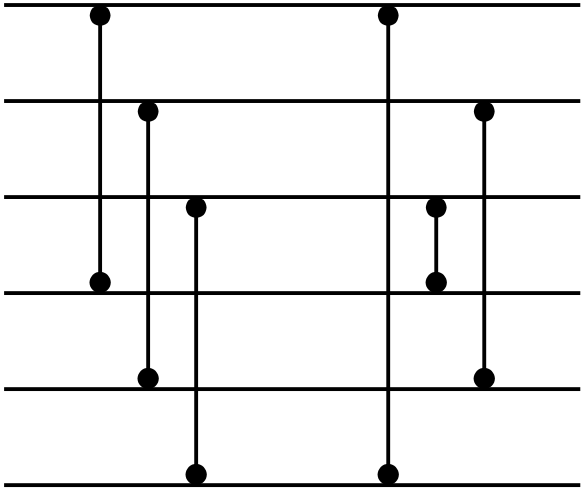
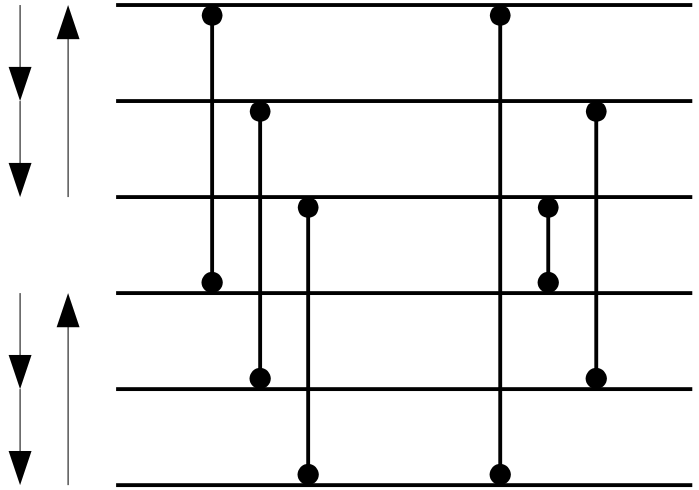
n	3	4	5	6	7	8	9	10	11	12	13	14	15	16
d	2	2	4	4	5	5	6	6	7	7	8	8	8	8
UNSAT	<1s	<1s	<1s	<1s	2s	2s	15m	29m	-	-	-	-	-	-
First Fixed	<1s	<1s	<1s	<1s	<1s	<1s	2s	10s	13m	23m	-	-	-	-
d	3	3	5	5	6	6	7	7	8	8	9	9	9	9
SAT	<1s	<1s	<1s	<1s	<1s	1s	1m	16m	21m	-	-	-	-	-
First Fixed	<1s	<1s	<1s	<1s	<1s	<1s	1s	19s	4m	12m	-	-	-	-

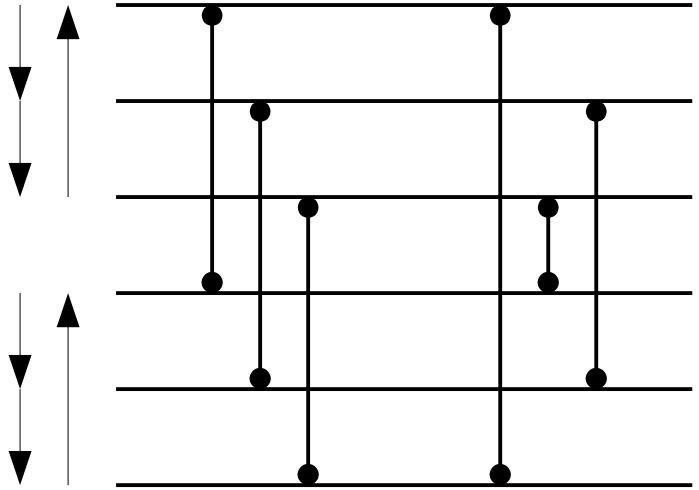
How can we restrict second layers for n=13?

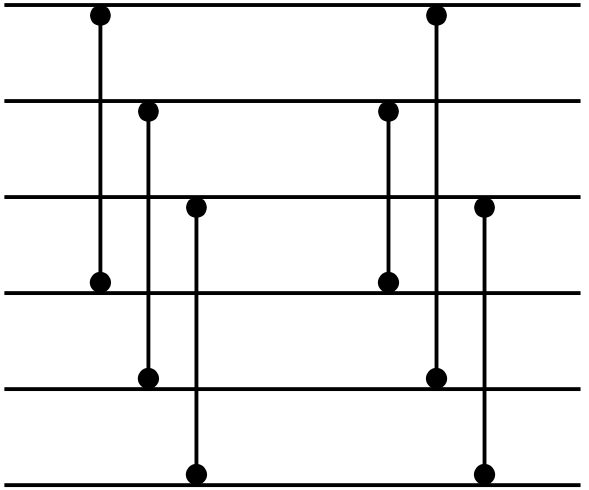


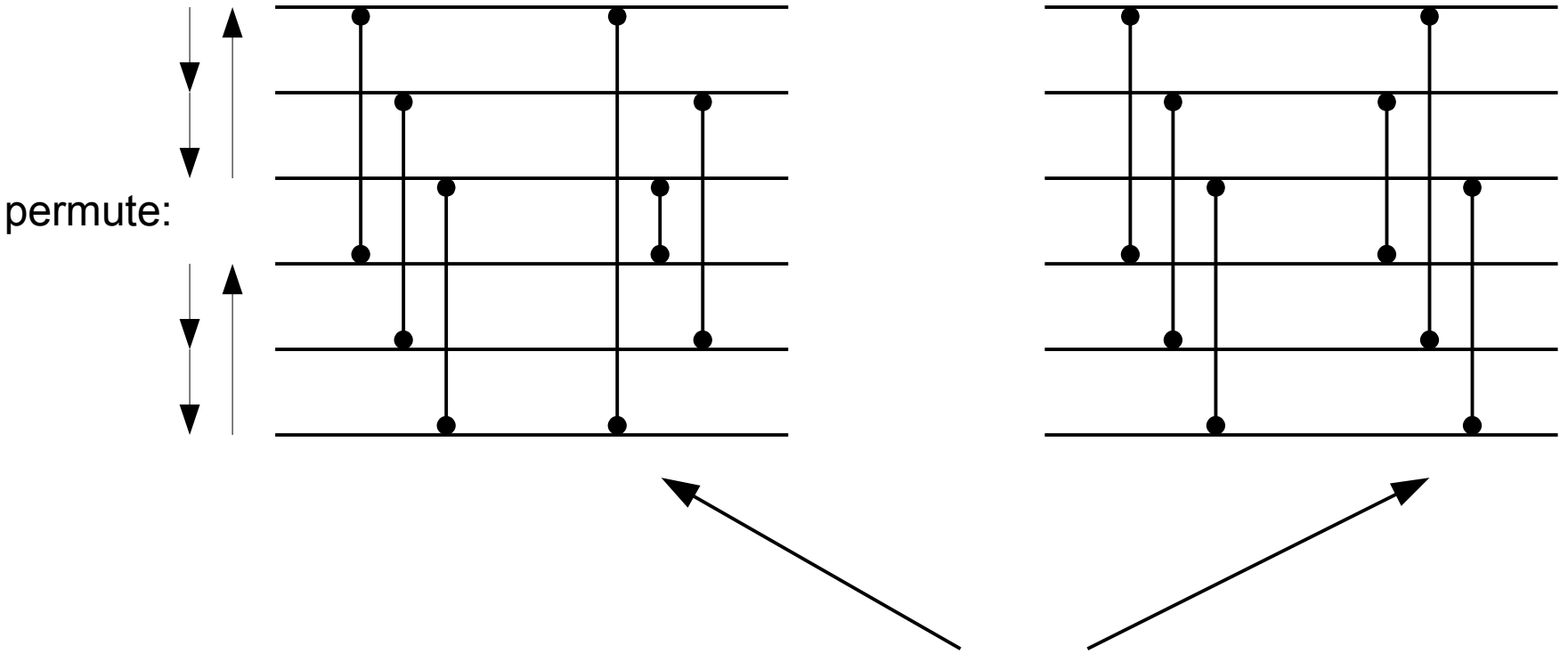
permute:

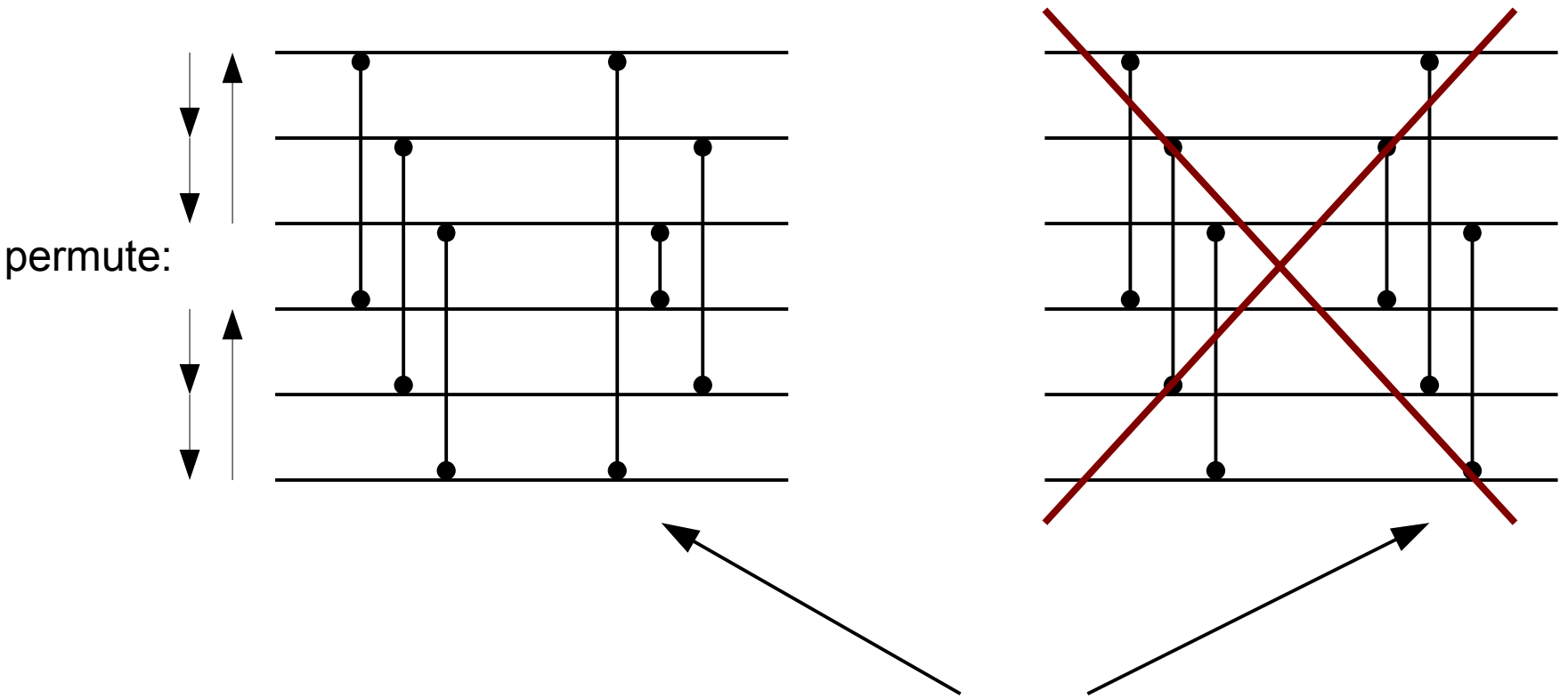


permute:

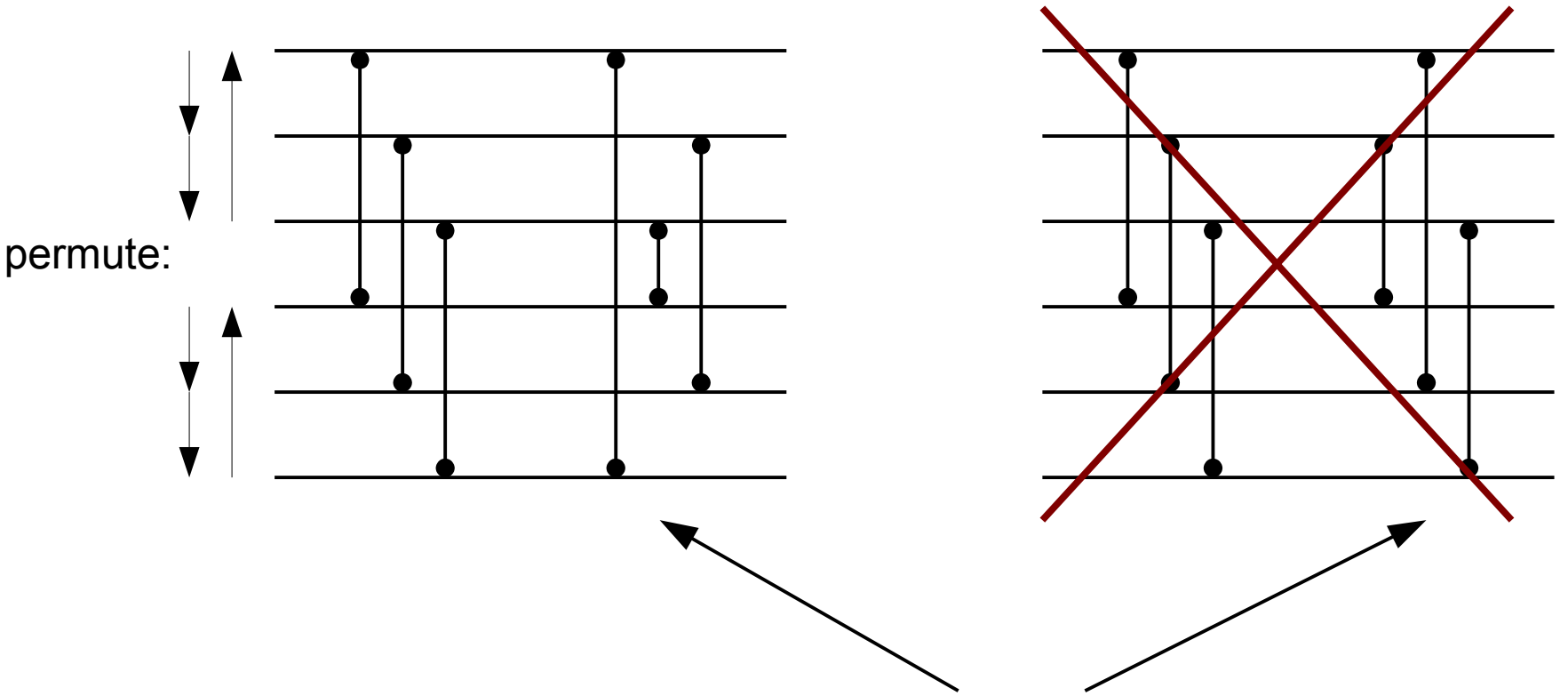





permute:
Equivalent second layers



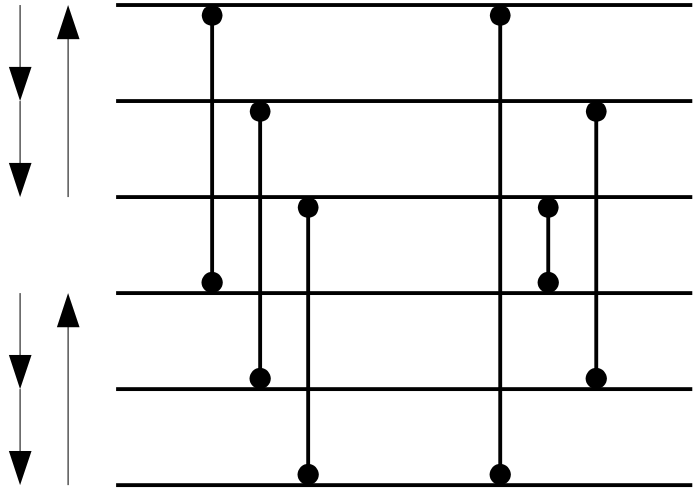
Equivalent second layers



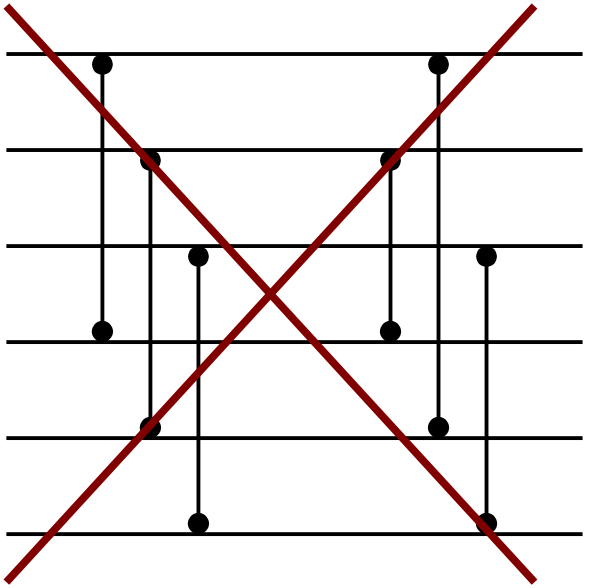
Equivalent second layers

For n=13 568504 candidate layers 4288 equivalence classes of second layers

permute:

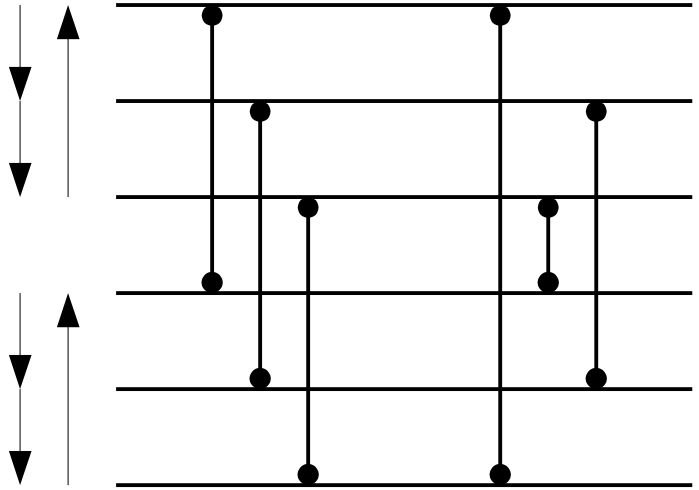


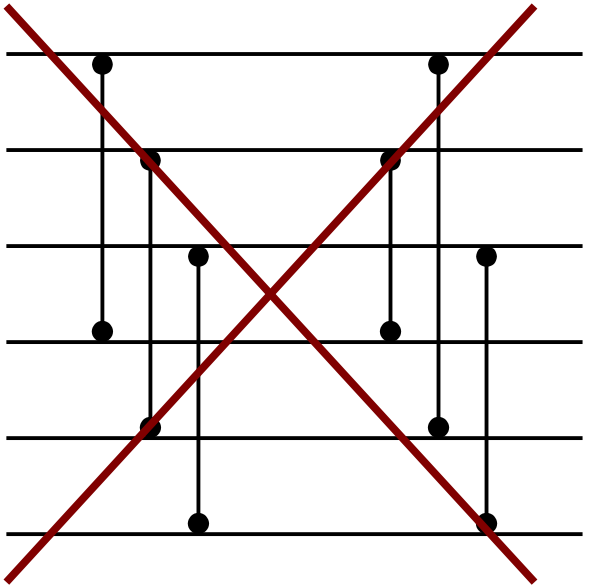
C = π(D)



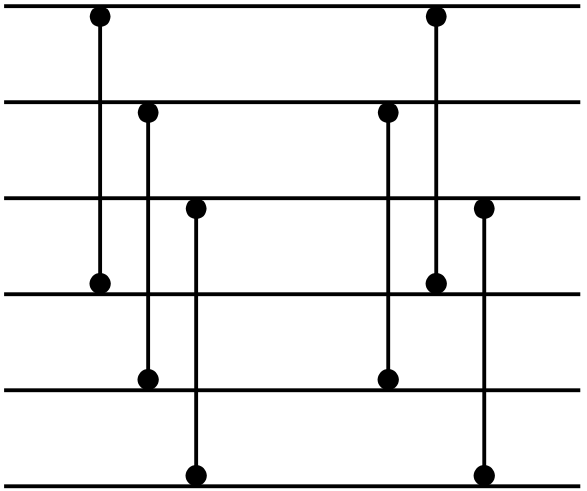
C = π(D)

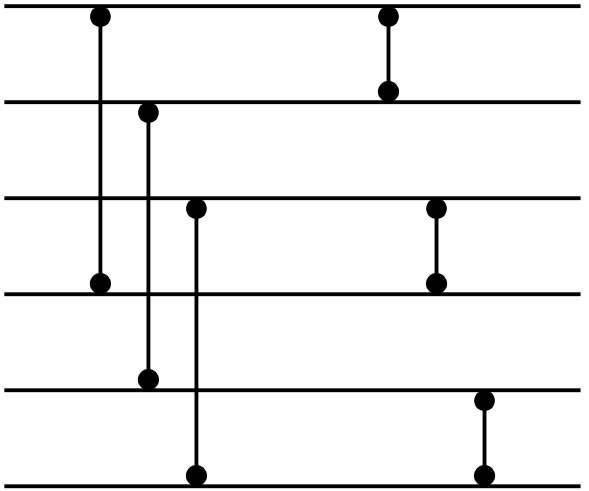
permute:



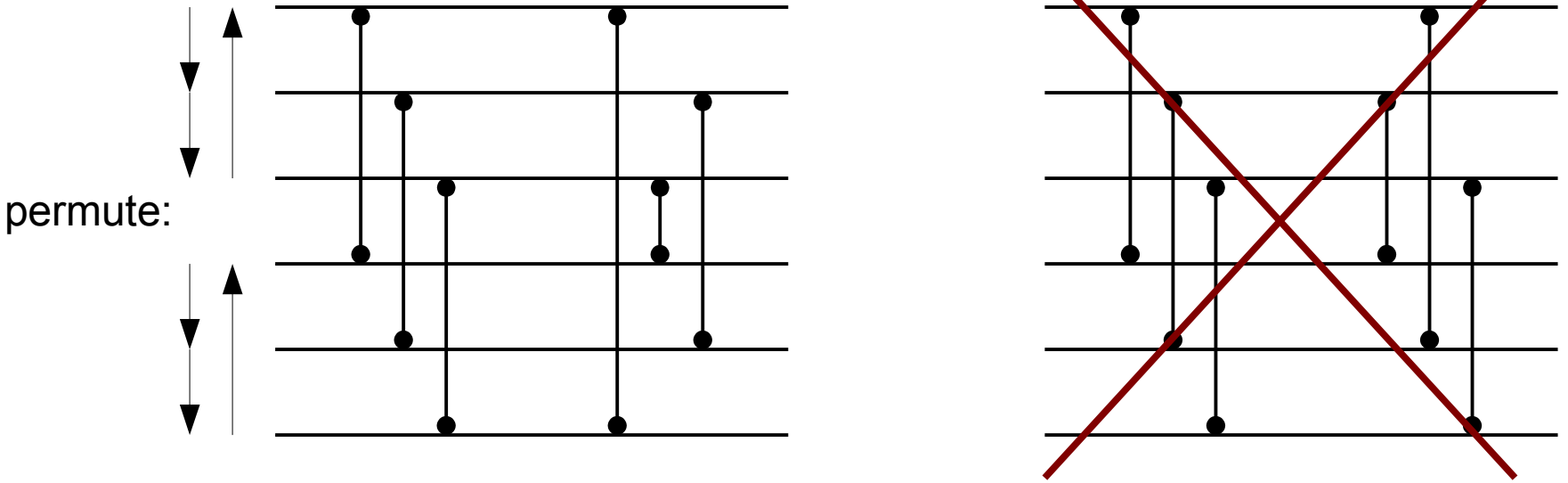


outputs(C) ⊆ outputs(D)

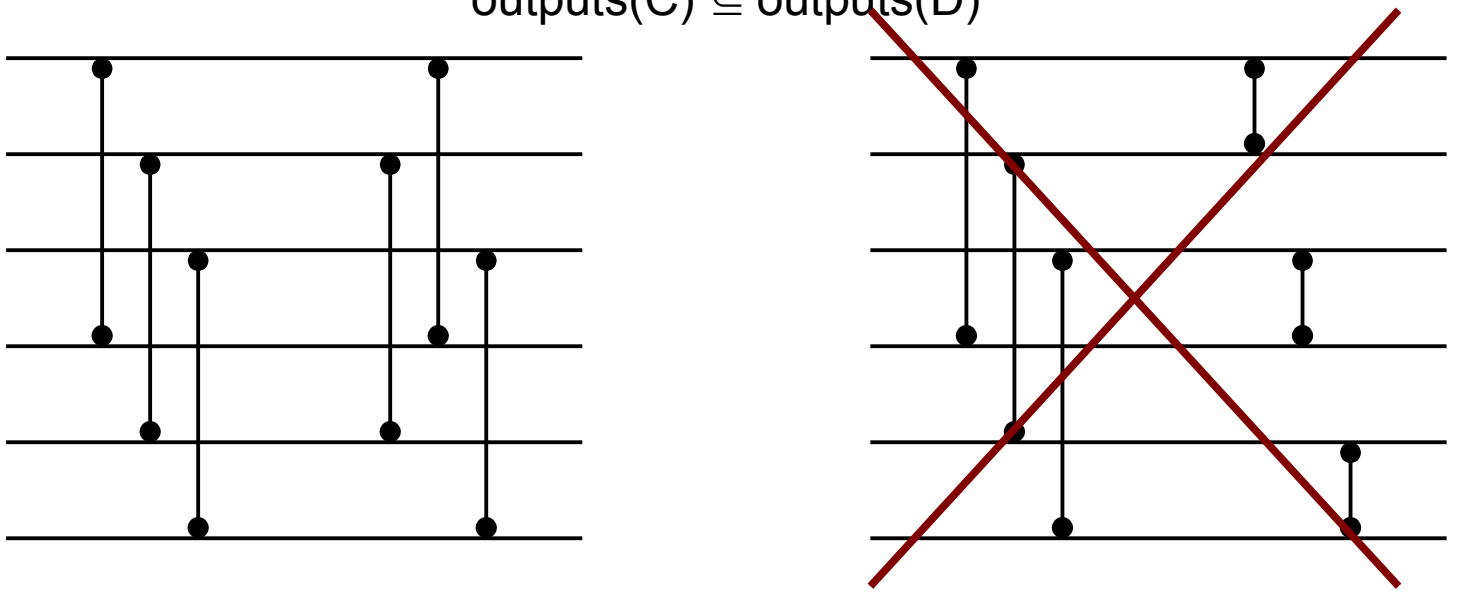




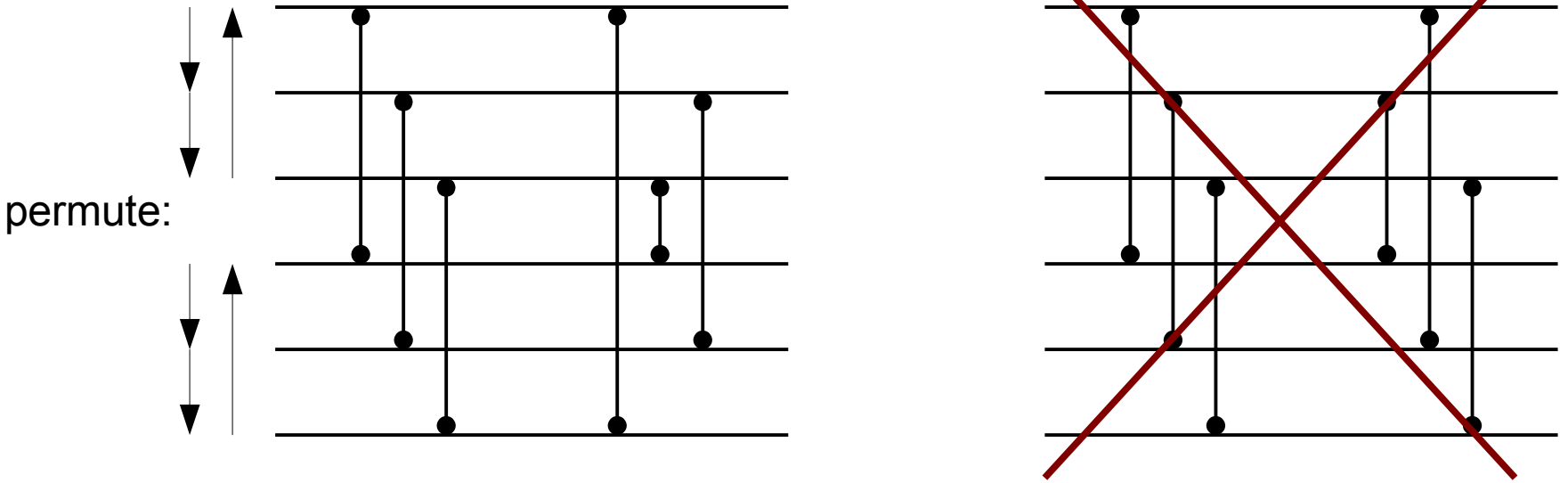
C = π(D)



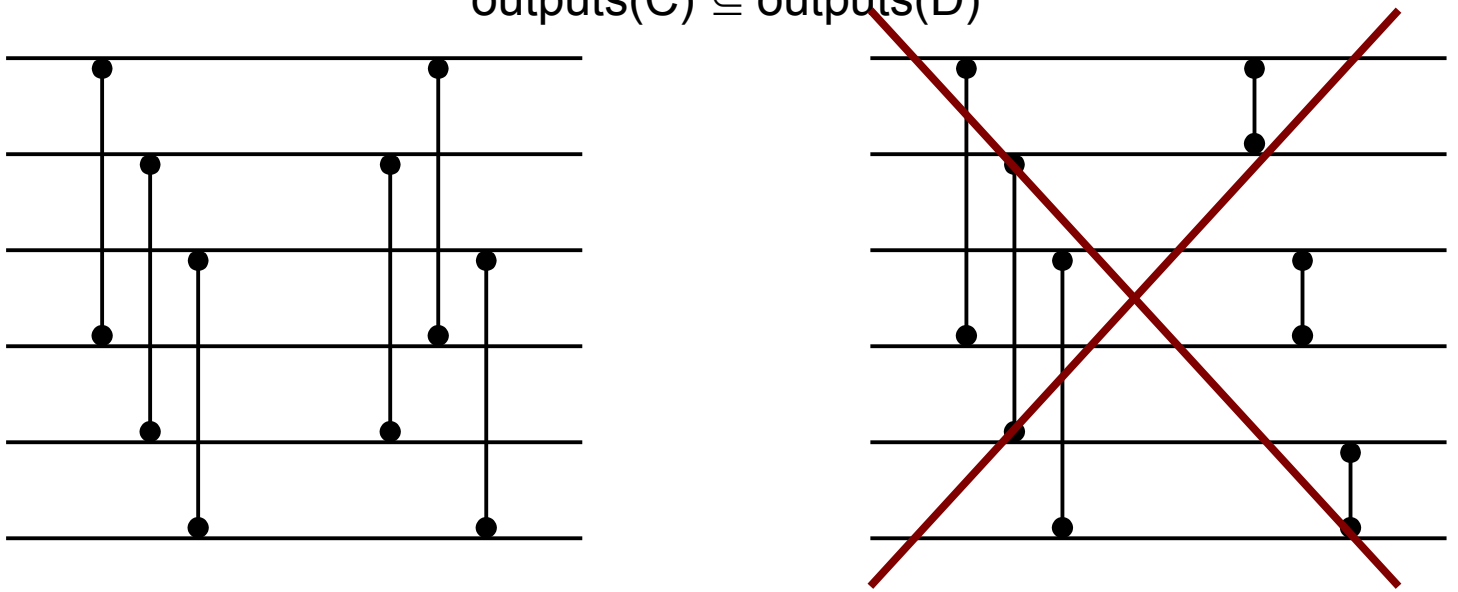
outputs(C) ⊆ outputs(D)



C = π(D)



outputs(C) ⊆ outputs(D)



outputs(C) ⊆ π(outputs(D))

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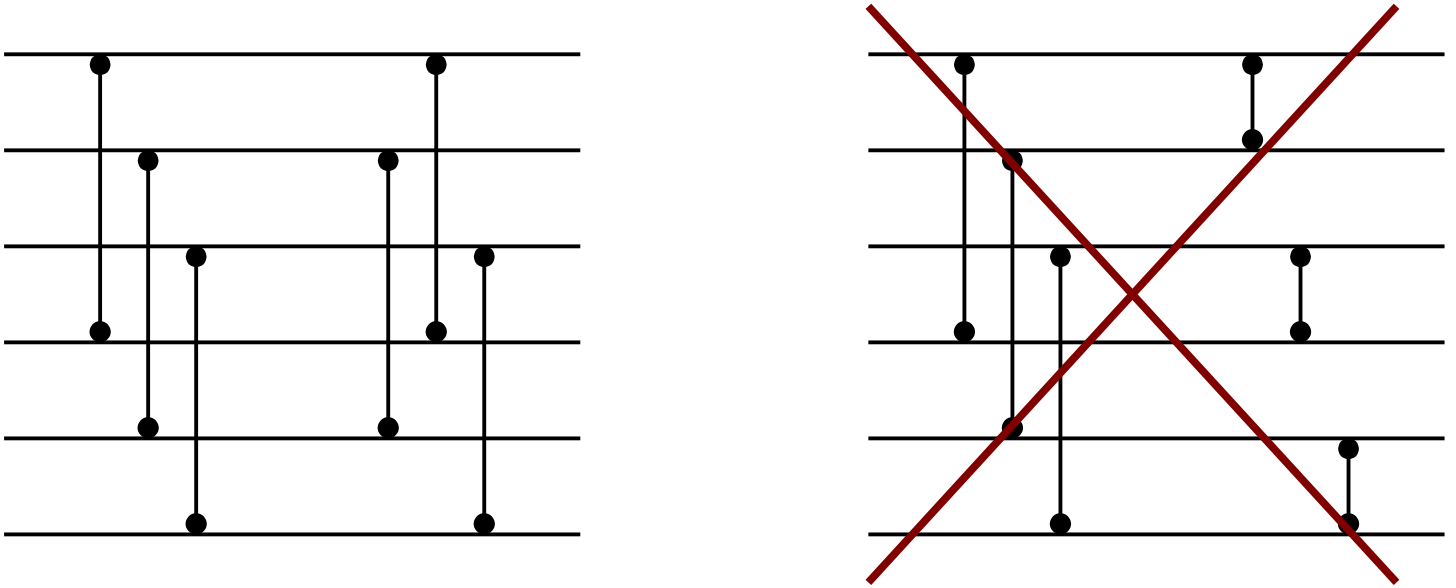

outputs(C) ⊆ π(outputs(D))

For n=13 568504 candidate layers

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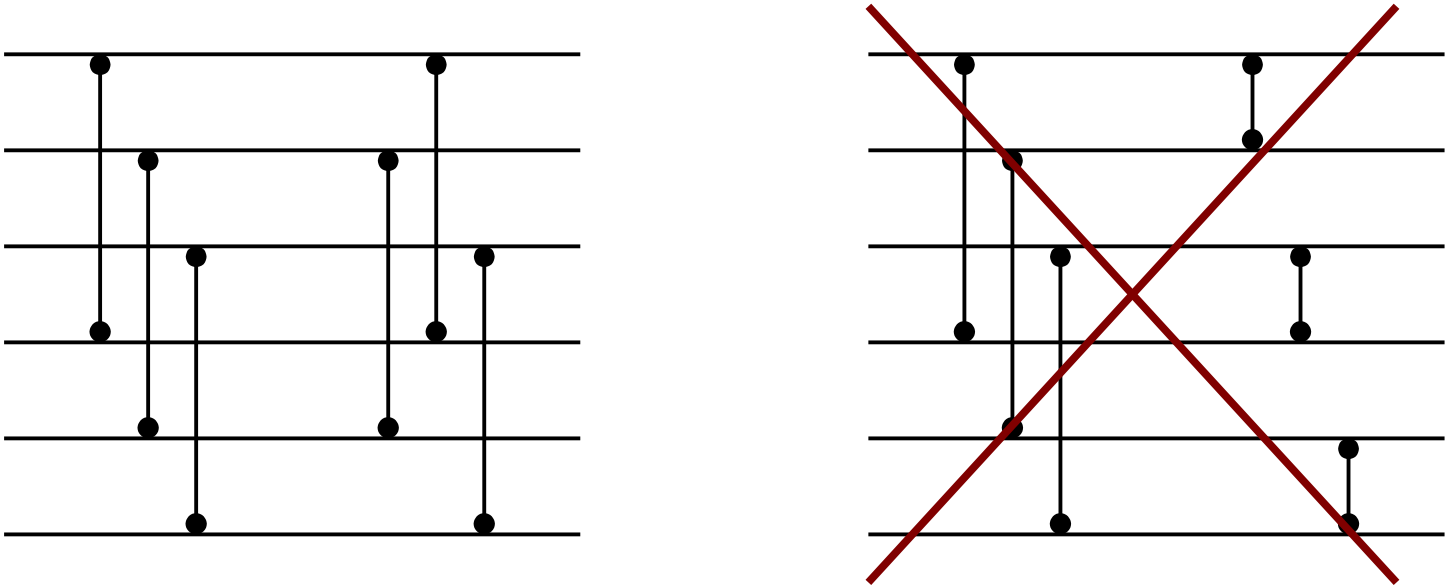
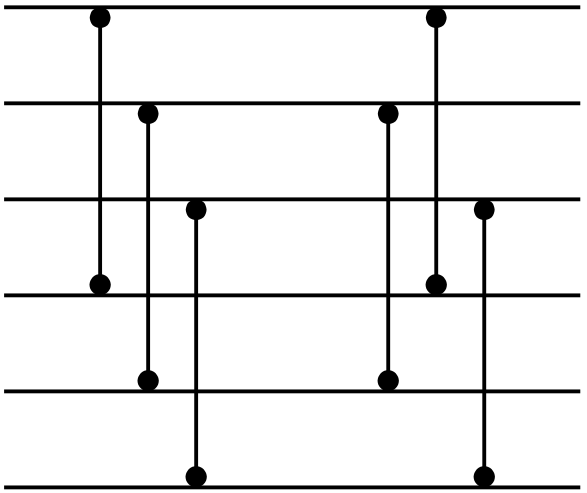


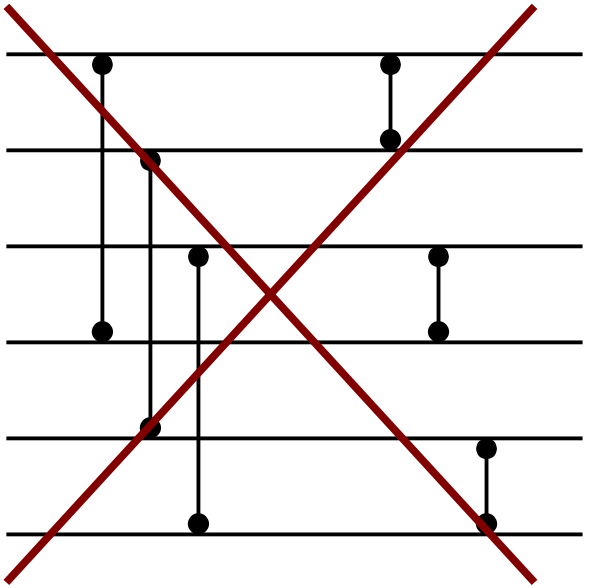
outputs(C) ⊆ π(outputs(D))

For n=13 568504 candidate layers

outputs(C) ⊆ π(outputs(D))

13! ~ 6×10^9 permutations





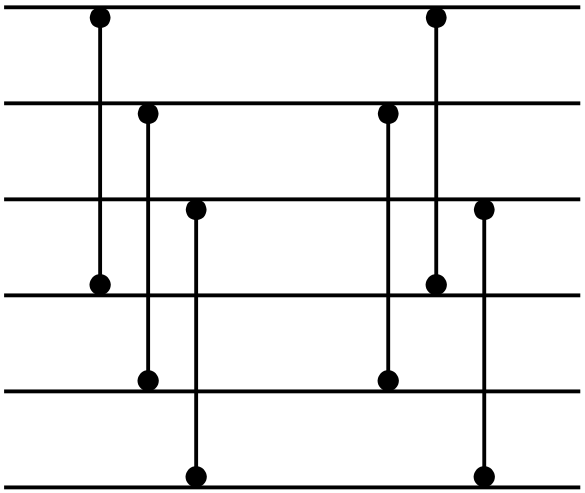
outputs(C) ⊆ π(outputs(D))

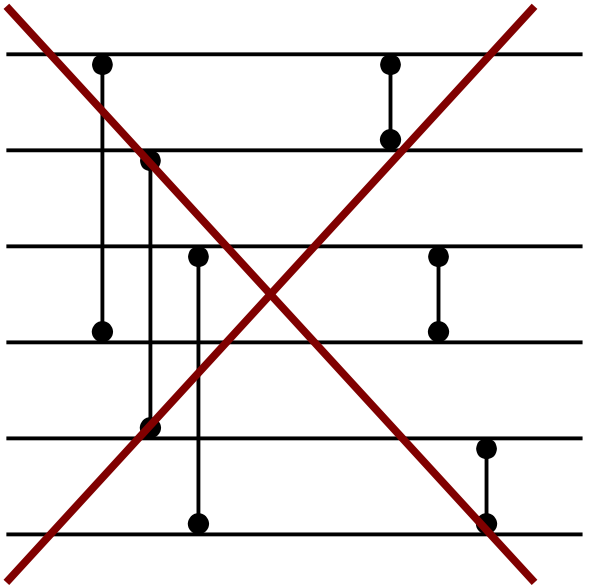
For n=13 568504 candidate layers

outputs(C) ⊆ π(outputs(D))

~1000 size of outputs(C)

13! ~ 6×10^9 permutations





outputs(C) ⊆ π(outputs(D))

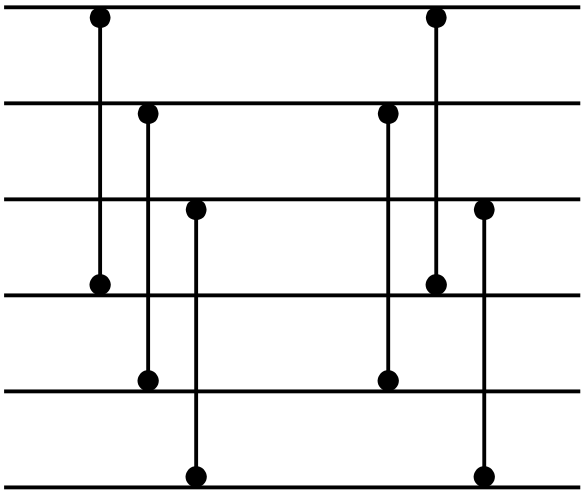
For n=13 568504 candidate layers

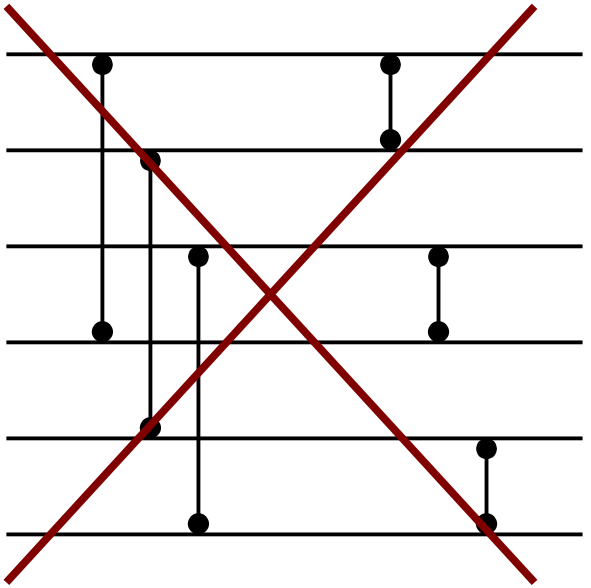
1069 minimal second layers
212 if you listen to Knuth
(118 if you are Knuth)

outputs(C) ⊆ π(outputs(D))

~1000 size of outputs(C)

13! ~ 6×10^9 permutations





outputs(C) ⊆ π(outputs(D))

For n=13 568504 candidate layers

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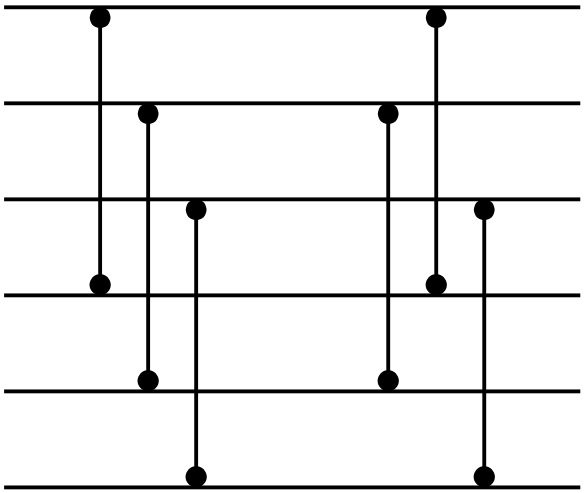

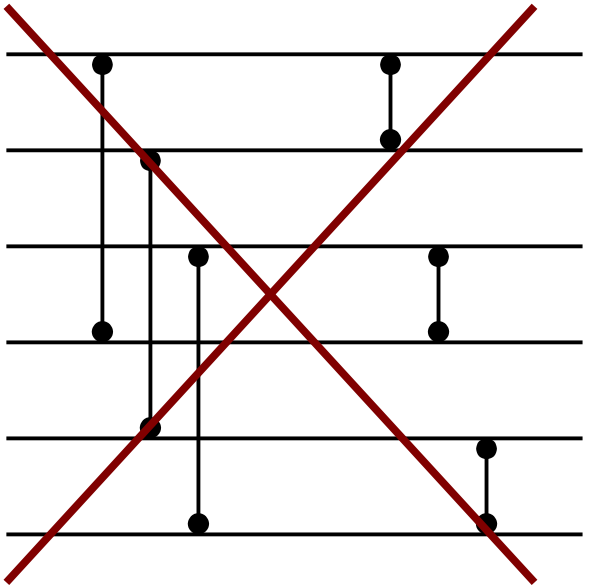

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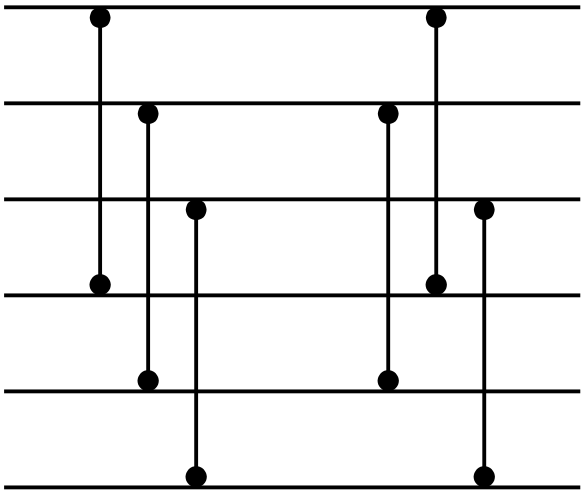
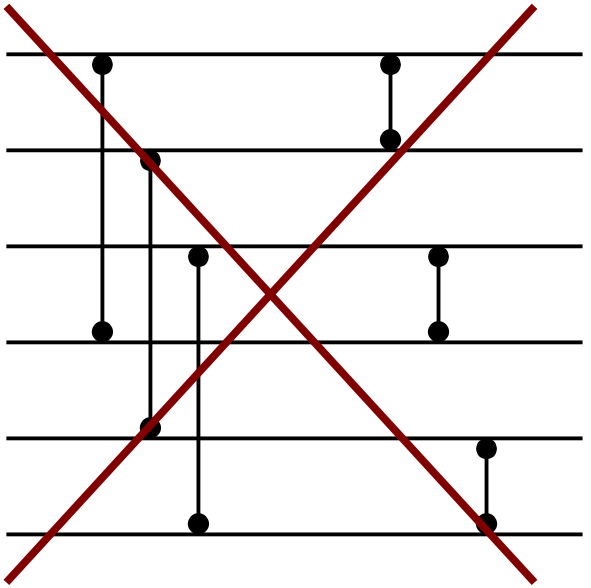


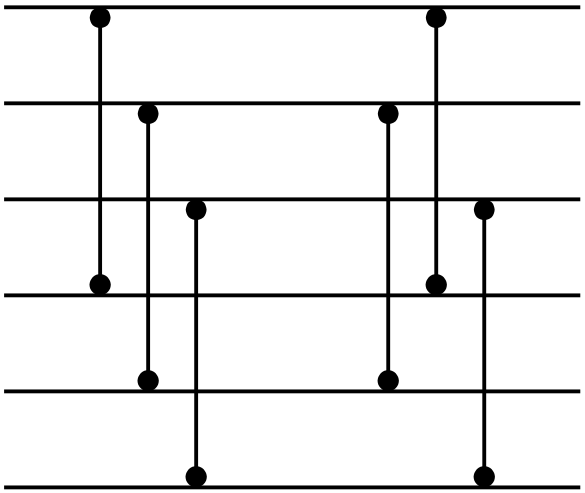


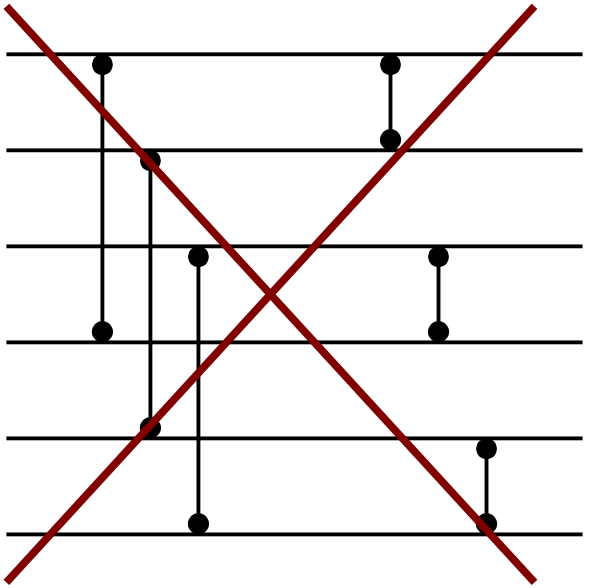
outputs(C) ⊆ π(outputs(D))

For n=13 568504 candidate layers 1069 minimal second layers
212 if you listen to Knuth
(118 if you are Knuth)

Goal: Show that no network of depth 8 sorts 13 inputs

Sufficient to fix the first layer and check 212 second layers





outputs(C) ⊆ π(outputs(D))

Goal: Show that no network of depth 8 sorts 13 inputs

Solution: run 212 instances of SAT (one for each second layer)

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Solution: run 212 instances of SAT (one for each second layer)

Network sorts every input:
sorts(00000000000) and sorts(00000000001) and sorts(00000000010) and ... and sorts(11111111111)

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Solution: run 212 instances of SAT (one for each second layer)

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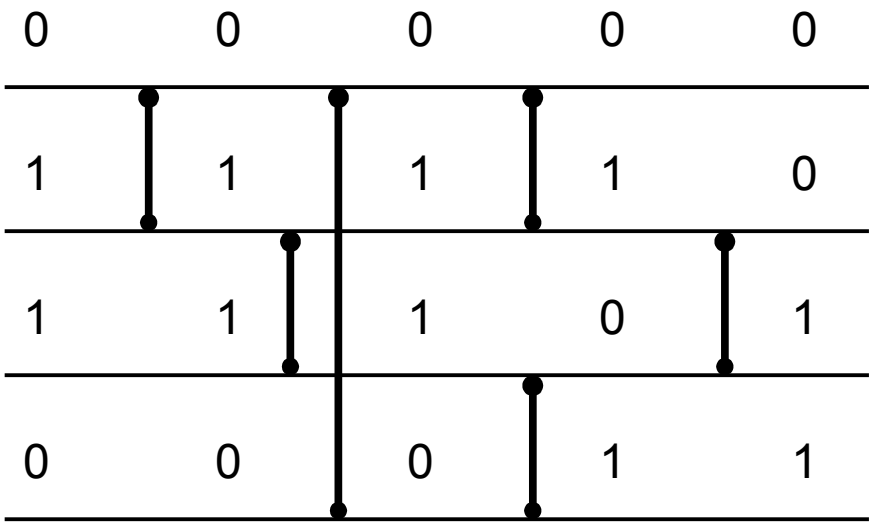

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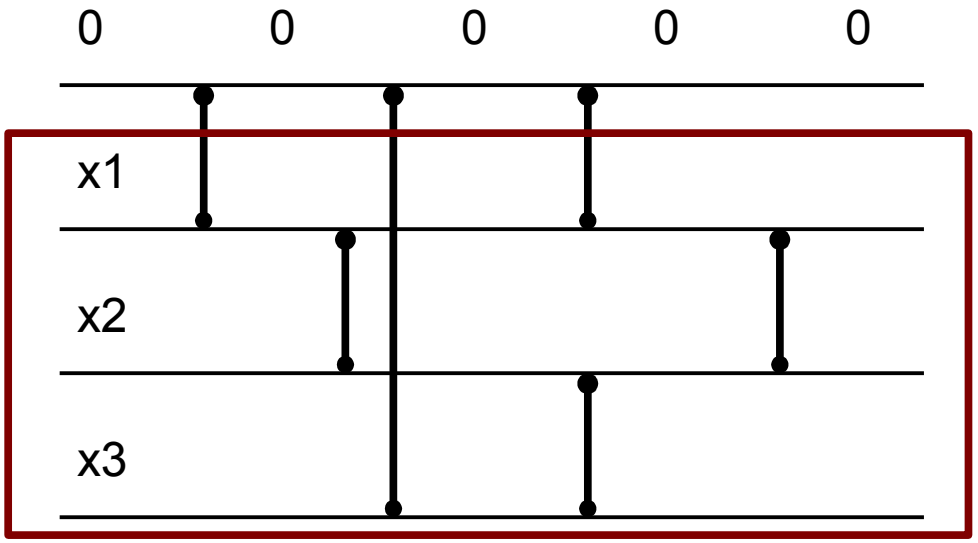
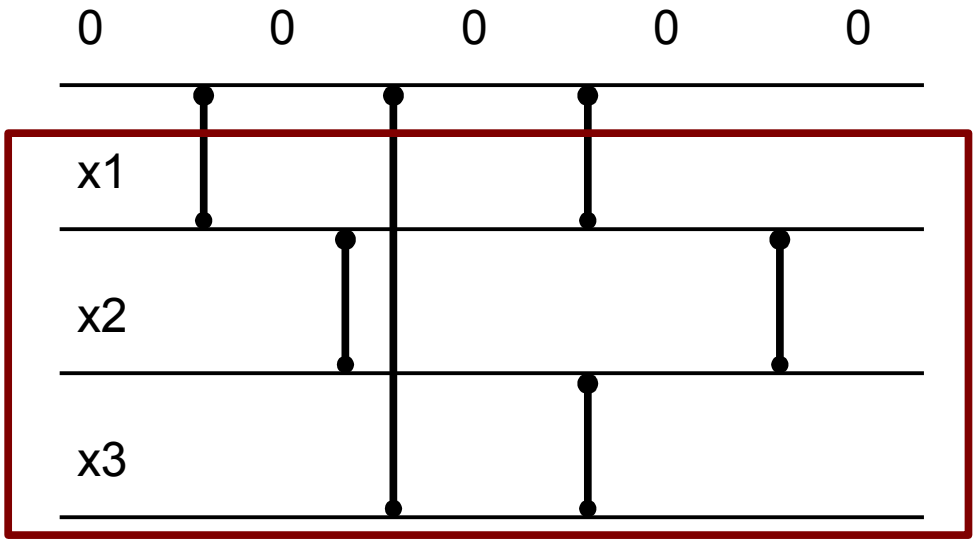
Network sorts every input:
sorts(00000000000) and sorts(00000000001) and sorts(00000000010) and ... and sorts(11111111111)



Goal: Show that no network of depth 8 sorts 13 inputs

Solution: run 212 instances of SAT (one for each second layer)

Network sorts every input:
sorts(00000000000) and sorts(00000000001) and sorts(00000000010) and ... and sorts(11111111111)

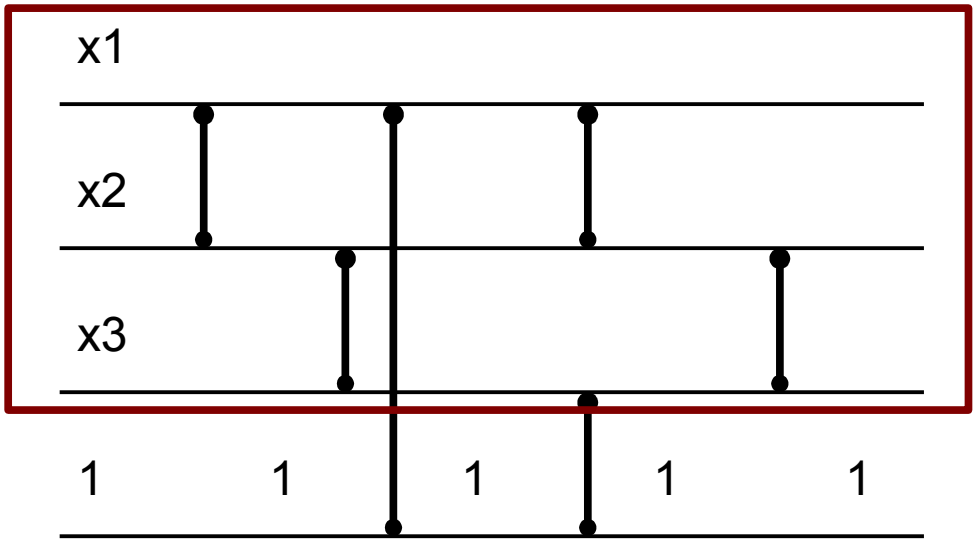


For n=13, try only inputs: 000x

Goal: Show that no network of depth 8 sorts 13 inputs

Solution: run 212 instances of SAT (one for each second layer)

Network sorts every input:
sorts(00000000000) and sorts(00000000001) and sorts(00000000010) and ... and sorts(11111111111)

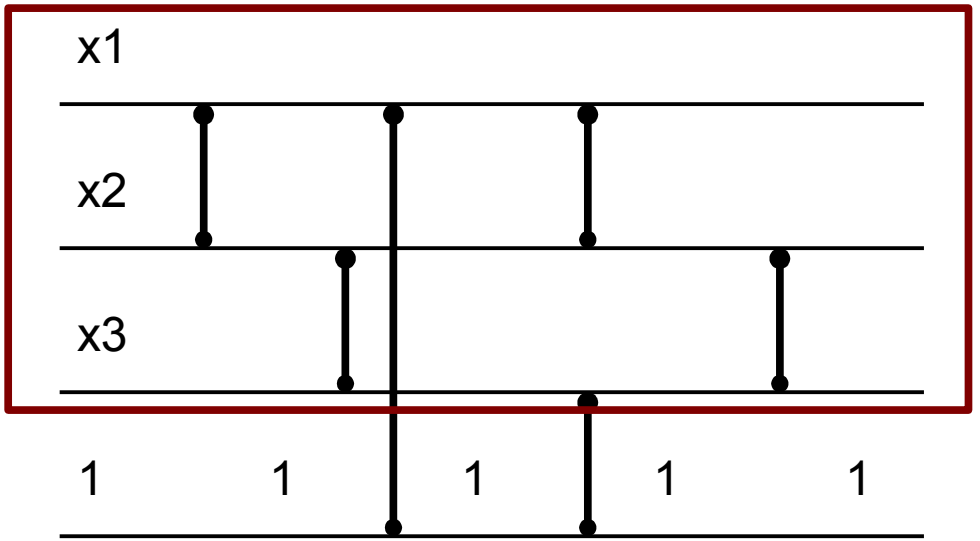


For n=13, try only inputs: 000x, y111, 0z11, 00w1

Goal: Show that no network of depth 8 sorts 13 inputs

Solution: run 212 instances of SAT (one for each second layer)

Network sorts every input:
sorts(00000000000) and sorts(00000000001) and sorts(00000000010) and ... and sorts(11111111111)



For n=13, try only inputs: 000x, y111, 0z11, 00w1

One instance: ~4min. Overall: 13 hours.

Goal: Show that no network of depth 8 sorts 13 inputs

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Network sorts every input:
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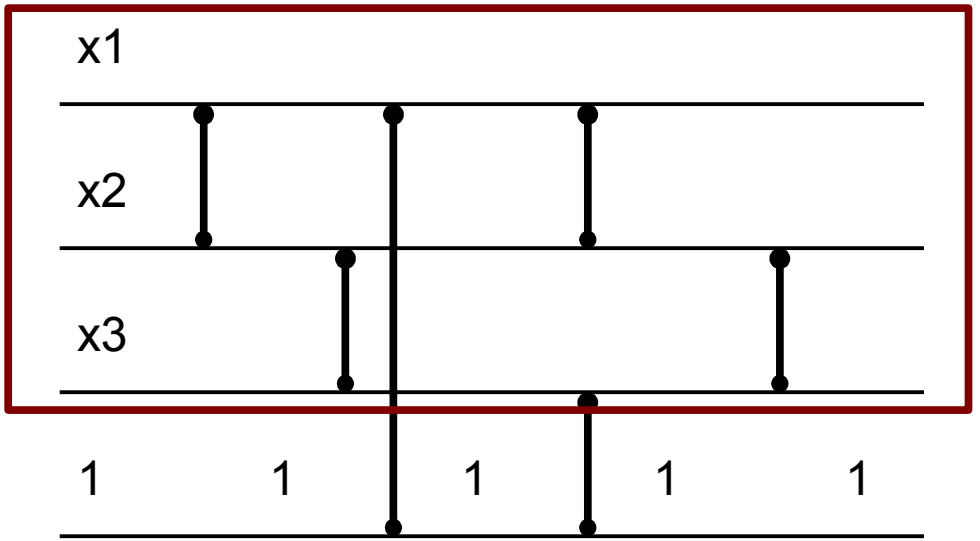
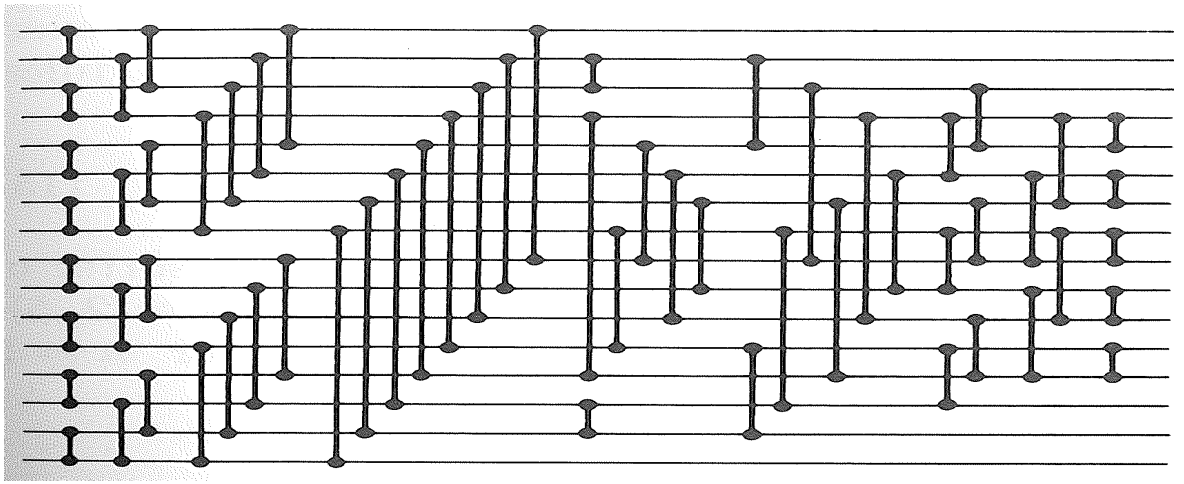


For n=13, try only inputs: 000x, y111, 0z11, 00w1

One instance: ~4min. Overall: 13 hours.

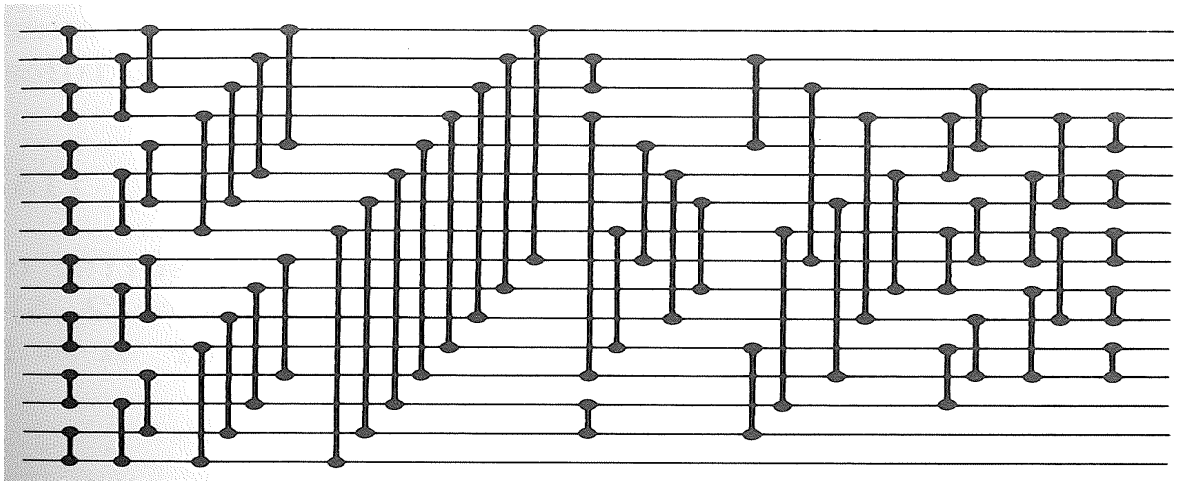
Optimal depth for 13 inputs is 9

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Upper Bound	0	1	3	3	5	5	6	6	7	7	8	8	9	9	9	9
Lower Bound	0	1	3	3	5	5	6	6	7	7	**8**	**8**	**9**	**9**	**9**	**9**



n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Upper Bound	0	1	3	3	5	5	6	6	7	7	8	8	9	9	9	9
Lower Bound	0	1	3	3	5	5	6	6	7	7	**8**	**8**	**9**	**9**	**9**	**9**

We reconstructed all previous results in 15 seconds



n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Upper Bound	0	1	3	3	5	5	6	6	7	7	8	8	9	9	9	9
Lower Bound	0	1	3	3	5	5	6	6	7	7	**8**	**8**	**9**	**9**	**9**	**9**

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