Probabilistic ω-Regular Expressions

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Motivation

Background

- Classical regular expressions in almost every field of theoretical computer science (Kleene 1956)
- Probabilistic automata on finite words well-studied with manifold applications (Rabin 1963)
- Regular Expressions transferred to probabilistic setting on finite words (Bollig, Gastin, Monmege, Zeitoum 2012)
- Probabilistic automata extended to infinite words (Baier, Grösser 2005)

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In this talk

1. Probabilistic regular expressions on infinite words expressively equivalent to probabilistic Muller-automata
2. "Probabilistic star-free expressions" with decidable emptiness and approximation problem

Probabilistic ω-Regular Expressions

Syntax

- Atomic expressions: • a (for $a \in \Sigma$) • p (for $p \in [0, 1]$)
- Compound expressions: • $E + F$ • $E \cdot F$ • E^* • E^ω

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Semantics

- $||a||(w) = \begin{cases} 1 & \text{if } w = a \\ 0 & \text{otherwise} \end{cases}$ and $||p||(w) = \begin{cases} p & \text{if } w = \varepsilon \\ 0 & \text{otherwise} \end{cases}$
- $||E + F||(w) = ||E||(w) + ||F||(w)$
- $||EF||(w) = \sum_{uv=w} ||E||(u)||F||(v)$
- $||E^*||(w) = \sum_{n \geq 0} ||E^n||(w)$ $(||E^*||(\varepsilon) = 1)$

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- $\left\|E^\omega\right\|(w) = \lim_{n\to\infty} ||E^n\Sigma^\omega||(w)$ $\quad (||\Sigma^\omega||(w) = 1)$

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- Compound expressions: • $E + F$ • $E \cdot F$ • E^* • E^ω
- Add special syntax restrictions

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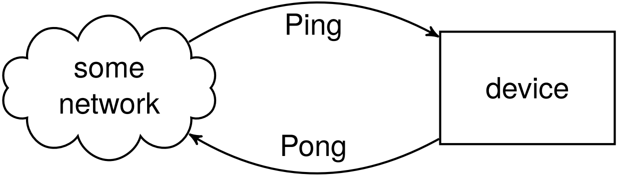
Definition

Set of probabilistic ω-regular expressions = smallest set $\mathcal{R}$ such that

1. $\Sigma^\omega \in \mathcal{R}$
2. $\sum_{a\in\Sigma} aE_a \in \mathcal{R}$ if $E_a \in \mathcal{R}$ for each $a \in \Sigma$
3. $pE + (1-p)F \in \mathcal{R}$ if $E, F \in \mathcal{R}$ and $p \in [0,1]$
4. $EF \in \mathcal{R}$ if $E\Sigma^\omega, F \in \mathcal{R}$
5. $E^*F + E^\omega \in \mathcal{R}$ if $E\Sigma^\omega + F \in \mathcal{R}$
6. $E \in \mathcal{R}$ if $E + F \in \mathcal{R}$
7. Close $\mathcal{R}$ under usual distributivity, associativity, commutativity

Example: Ping Pong



- Network device, which responds to “Ping” messages
- Pong should be sent before next Ping
- Input = Sequence of “**p**ing request” or “**n**othing” $\rightsquigarrow \Sigma = \{p, n\}$

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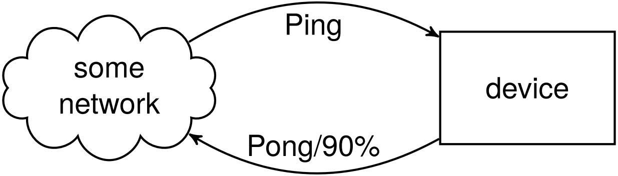

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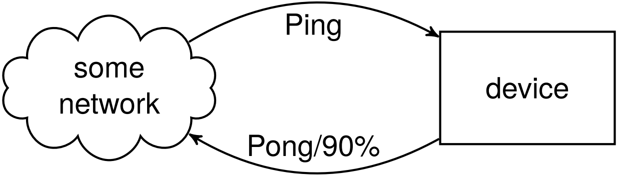

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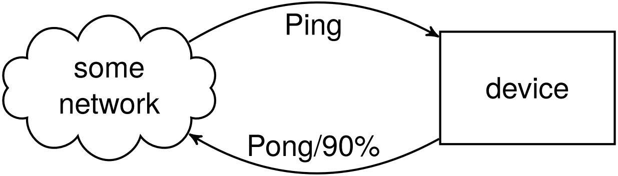

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$$E = \left(n^* p \left(\frac{1}{10} n\right)^* \frac{9}{10} n\right)^\omega$$

- $\|E\|(uv^\omega) = 0$ for all $u, v \in \Sigma^+$ with $v \notin \{n\}^+$
- $\|E\|(pnpn^2pn^3p \dots) > 0$

Expressive Equivalence

Theorem

Let $f : \Sigma^\omega \to [0,1]$. TFAE:

1. $f = \|A\|$ *for some probabilistic Muller-automaton* A
2. $f = \|E\|$ *for some probabilistic* ω*-regular expression* E

Idea of the Proof

Expression → Automaton

- Inductive construction on syntax of expression
- Based on ideas of the finite word case
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Automaton → Expression

- Use induction on $|X|$ for set $X \subseteq Q$ to build expressions E_p^X:

$$E_p^X = \sum_{q \notin X} E_{p,q}^X \Sigma^\omega + \sum_{F \subseteq X} E_{p,\mathrm{inf}=F}^X$$

- In induction step:
  Consider prefix-minimal runs visiting all states in X in fixed order

Decidability Results

- Effective transformations: automata $\leftrightarrow$ expressions
- Automata-based decidability results transfer to expressions

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- Given expression E, $\exists u, v \in \Sigma^+ : \|E\|(uv^\omega) > 0$?

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- Effective transformations: automata $\leftrightarrow$ expressions
- Automata-based decidability results transfer to expressions

Decidable:

- Given expression E, $\exists u, v \in \Sigma^+ \colon \|E\|(uv^\omega) > 0$?

Undecidable:

- Given expression E, $\exists w \in \Sigma^\omega \colon \|E\|(w) > 0$ $(= 1)$?
- Given expression E and $\varepsilon > 0$ such that either
  1. $\forall w \in \Sigma^\omega \colon \|E\|(w) \leq \varepsilon$
  2. $\exists w \in \Sigma^\omega \colon \|E\|(w) \geq 1 - \varepsilon$

  Which is the case?

A “Probabilistic Star-Free” Fragment of Expressions

- Restrict probabilistic iteration,
  such that iteration almost surely terminates
- Allow only the following iteration constructs
  1. E^* and E^ω for deterministic expression E
  2. $(pE)^*$ for $p < 1$ and expression E
  - Nesting 1 within 2 (2 within 1) not allowed

$\rightsquigarrow$ *Almost ω-deterministic expressions*

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- $\left(1/3 \cdot (2/3 \cdot a)^* a\right)^* b^\omega$ Not an expression

Almost Limit-Deterministic Automata

Definition

$A = (Q, \delta, \mu, \mathrm{Acc})$ *almost limit-deterministic* if for every SCC $C \subseteq Q$

$$\delta(p, a, q) \in \{0, 1\} \text{ for all } p, q \in C,\ a \in \Sigma$$

or $\Pr^{w}_{A_q}(C^{\omega}) = 0$ for all $q \in C$ and $w \in \Sigma^{\omega}$

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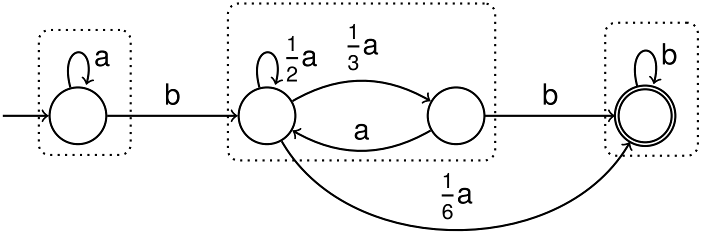

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Remark

- Only countable many runs in ALD automata
  - $\rightsquigarrow$ Decidable positive emptiness problem
- Class of ALD automata closed under cross-product, complement, concatenation
- Cannot express infinitely many probabilistic choices

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Theorem

Let A be almost limit deterministic and $\varepsilon > 0$. Then

$$\exists \text{ finite, computable } V \subseteq [0, 1] \colon d_{\mathrm{H}}(\|A\|(\Sigma^{\omega}), V) \leq \varepsilon,$$

where d_{H} is Hausdorff-distance, i.e.

- $\forall w \in \Sigma^{\omega} \colon \exists x \in V \colon |x - \|A\|(w)| \leq \varepsilon$
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Theorem

Let E almost ω-deterministic expression.
There is almost limit deterministic automaton A s.t. $\|A\| = \|E\|$.

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Future research

- Applications for almost limit-deterministic automata
- Characterization of ALD automata by "probabilistic FO" logic
- Add weights to the expression framework
  - Probabilistic weighted A. by Chattergee, Doyen, Henzinger
  - Valuation monoids by Droste, et.al.