Solutions to the multi-dimensional equal powers problem constructed by composition of rectangular morphisms

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Basics

- $\mathbb{N} = \{0, 1, 2, \ldots\}$
- for $n \in \mathbb{N}$, $[\![n]\!] = \{0, 1, \ldots, n-1\}$
- *multiset* on a set U is a mapping $A : U \to \mathbb{N}$
- $(\Sigma^*, \cdot, \lambda)$ - free monoid
- *morphism* - $h : \Sigma^* \to \Gamma^*$, $h(\alpha\beta) = h(\alpha)(\beta)$
  - *uniform* - $|h(a)| = |h(b)|$ for $a, b \in \Sigma$

Subwords

- *factor* u $\qquad w = tuv$
- *subword* - subsequence
- $|\alpha|_u$ -number of occurrences of the subword u in α.
  e.g.,
  - the subword occurring at $\{1, 2, 5, 6\}$ in *babbaba* is *abba*

0	1	2	3	4	5	6
b	*a*	*b*	*b*	*a*	*b*	*a*

  - the set of all occurrences of *abba* in *babbaba* is $\{\{1, 2, 3, 4\}, \{1, 2, 3, 6\}, \{1, 2, 5, 6\}, \{1, 3, 5, 6\}\}$
  - $|babbaba|_{abba} = 4$

History

1851 Eugène Prouhet

1906 Axel Thue

1921 Marston Morse

1929 Max Euwe

The PTE Problem

Prouhet-Tarry-Escott (*PTE*) problem
More general version

Problem ($PTE_{n,k}$)

Find n equally-sized sets of integers, satisfying pairwise

$$\sum_{a\in A} a^s = \sum_{b\in B} b^s \text{ for all } s < k.$$

Prouhet's words and Thue morphism

$$\alpha_{\langle n,k \rangle} = h^k(0) \text{, where } \begin{array}{l} h : [\![n]\!]^* \to [\![n]\!]^* \\ h(0) = 01 \dots \langle n-1 \rangle \\ h(1) = 12 \dots \langle n-1 \rangle\, 0 \\ \dots \end{array}$$

Groups - positions of $0, 1, \dots, \langle n-1 \rangle$ in $\alpha_{\langle n,k \rangle}$

Example

$n = 2, k = 4$

$$h : \{0,1\}^* \to \{0,1\}^*, h(0) = 01, h(1) = 10$$
$$\alpha_{\langle 2,4 \rangle} = h^4(0) = 0110100110010110$$

Groups:

	$0:$	$0, 3, 5, 6, 9, 10, 12, 15$
	$1:$	$1, 2, 4, 7, 8, 11, 13, 14$

The sum of first powers is 60, the sum of second powers is 620 and the sum of third powers is 7200.

Prouhet's Theorem

Theorem (Prouhet 1851)

If the integers $0, ..., n^k - 1$ *are partitioned into* n *parts* $(E_0, \ldots, E_{n-1})$ *such that the sum* $(\bmod n)$ *of the base-*n *digits of numbers in* E_z *equals* z*, then the sums of the* s*-th powers,* $0 \leq s < k$*, in each class are equal.*

Theorem (Lehmer 1947)

Let $q_0, q_1, \ldots, q_{k-1}$ *be any integers. If the integers* $0, ..., n^k - 1$ *are partitioned into* n *parts* E_z *such that the sum* $(\bmod n)$ *of the base-*n *digits of numbers in* E_z *equals* z*. If* $0 \leq s < k$*, then*

$$\sum_{a \in E_z} \left(\sum_{i=0}^{k-1} a_i q_i \right)^s$$

where $(a_0, a_1, \ldots, a_{n-1})$ *is the base-*n *notation of* a*, does not depend on* z*.*

Multi-dimensional case

Problem ($PTE^d_{n,k}$ - Alpers, Tijdeman, 2007)

For a given integer $d \geq 1$ (the dimension of the problem), $k \geq 1$ (the degree of the problem), and $n \geq 2$ find n sets of integer d-tuples satisfying pairwise, for all $0 \leq r_0, \ldots r_{d-1}$ such that $r_0 + \cdots + r_{d-1} < k$, the equality

$$\sum_{\langle a_0,\ldots,a_{d-1}\rangle \in A} a_1^{r_0} \cdot \cdots \cdot a_2^{r_{d-1}} \\ = \sum_{\langle b_0,\ldots,b_{d-1}\rangle \in B} b_1^{r_0} \cdot \cdots \cdot b_2^{r_{d-1}}$$

Remark

Our considerations will deal with the case $d = 2$.
Generalization to the case $d > 2$ will be straightforward.

Array words

$\Sigma = \{0,1\}$; $\langle 3,2\rangle$-word $\alpha = \begin{matrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{matrix}$

$\langle 2,2\rangle$-morphism $h : \{0,1\}^{**} \to \{0,1,2\}^{**}$ given as

$$h(0) = \begin{matrix} 0 & 1 \\ 2 & 0 \end{matrix} \qquad h(1) = \begin{matrix} 1 & 2 \\ 0 & 1 \end{matrix} \qquad .$$

Then $h(\alpha)$ is the $\langle 6,4\rangle$-word

$h(\alpha) =$

	0	1	2	3
0	0	1	1	2
1	2	0	0	1
2	1	2	1	2
3	0	1	0	1
4	1	2	0	1
5	0	1	2	0

Blue symbols:
subword determined by
rows 1,3,4
columns 1,3

$$\begin{matrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{matrix}$$

Spectra of array words

k-spectrum of a word $\alpha \in \Sigma^{m,n}$
multiset of array words

$$S_{\alpha}^{(k)} = \cup_{p+q \leq k+1} D_{\alpha}^{\langle p,q \rangle},$$

where $D_{\alpha}^{\langle p,q \rangle}$ is the $\langle p, q \rangle$-*deck* of α,
multiset on $\Sigma^{p,q}$, satisfying

$$D_{\alpha}^{(p,q)}(\gamma) = |\alpha|_{\gamma}.$$

Equal spectra of array words and the *PTE* problem

Theorem

Let $k, m, n \in \mathbb{N}$, $c \in \Sigma$, *and* $\alpha, \beta \in \Sigma^{m,n}$, *such that* $S_\alpha^{(k)} = S_\beta^{(k)}$. *Then, for each* $c \in \Sigma$,

$$\sum_{\substack{a_0 \in [\![m]\!], a_1 \in [\![n]\!] \\ \alpha_{a_0,a_1} = c}} a_0^r a_1^s = \sum_{\substack{b_0 \in [\![m]\!], b_1 \in [\![n]\!] \\ \beta_{b_0,b_1} = c}} b_0^r b_1^s.$$

for all $0 \leq r, s$ *such that* $r + s < k$.

Lemma

Let $\alpha, \beta \in \Sigma^{m,n}$*, and let* $p \leq m, q \leq n$*. If* $D_{\alpha}^{(p,q)} = D_{\beta}^{(p,q)}$ *then, for each* $c \in \Sigma$*,and* $r < p, s < q$

$$\sum_{\substack{a_0 \in [\![m]\!], a_1 \in [\![n]\!] \\ \alpha_{a_0, a_1} = c}} a_0^r a_1^s = \sum_{\substack{b_0 \in [\![m]\!], b_1 \in [\![n]\!] \\ \beta_{b_0, b_1} = c}} b_0^r b_1^s.$$

Proof.

Express the polynomial $x^r y^s$ as a linear combination of the polynomials $f_{i,j}^{m,n,p,q}(x, y)$. □

Equal spectra of array words and the *PTE* problem

The number of array words in $D_\alpha^{(p,q)}$(taking in consideration the multiplicity of their occurrences in α) containing the symbol $c \in \Sigma$ at position $\langle i, j \rangle$ is

$$\sum_{\substack{\langle t, < m, u < n \rangle \\ \alpha_{t,u} = c}} \binom{t}{i} \binom{m-t-1}{p-i-1} \binom{u}{j} \binom{n-u-1}{q-j-1}$$

Lemma

The polynomials

$$f_{i,j}^{m,n,p,q}(x, y) = \binom{x}{i} \binom{m-x-1}{p-i-1} \binom{y}{j} \binom{n-y-1}{q-j-1}$$

form a base of the vector space of all bivariate polynomials of bi-order $\langle p, q \rangle$.

Equal spectra of array words and the *PTE* problem

Assume $D_{\alpha}^{(p,q)} = D_{\beta}^{(p,q)}$

Denote

$$\delta_{t,u}^{c} = \begin{matrix} 1 & \text{if } \alpha_{t,u} = c \text{ and } \beta_{t,u} \neq c \\ -1 & \text{if } \alpha_{t,u} \neq c \text{ and } \beta_{t,u} = c \\ 0 & \text{if } \alpha_{t,u} = \beta_{t,u} = c \end{matrix}$$

Then

$$\sum_{t<m,u<n} \delta_{t,u}^{c} f_{i,j}^{m,n,p,q}(t,u) = \sum_{t<m,u<n} \delta_{t,u}^{c} \binom{t}{i} \binom{m-t-1}{p-i-1} \binom{u}{j} \binom{n-u-1}{q-j-1}$$
$$= 0$$

Equal spectra and the PTE problem

Lemma

Let $\alpha, \beta \in \Sigma^{m,n}$*, and let* $p \leq m, q \leq n$*. If* $D_{\alpha}^{(p,q)} = D_{\beta}^{(p,q)}$ *then, for each* $c \in \Sigma$*,and* $r < p, s < q$

$$\sum_{\substack{a_0 \in [\![m]\!], a_1 \in [\![n]\!] \\ \alpha_{a_0,a_1} = c}} a_0^r a_1^s = \sum_{\substack{b_0 \in [\![m]\!], b_1 \in [\![n]\!] \\ \beta_{b_0,b_1} = c}} b_0^r b_1^s.$$

Proof.

Express the polynomial $x^r y^s$ as a linear combination of the polynomials $f_{i,j}^{m,n,p,q}(x, y)$. □

Equal spectra and the PTE problem

$$x^r y^s = \sum_{i<p,j<q} \mu_{i,j} f_{i,j}^{m,n,p,q}(x,y)$$

Assume $D_\alpha^{(p,q)} = D_\beta^{(p,q)}$. Then

$$\sum_{\substack{t<m,u<n \\ \alpha_{t,u}=c}} t^r u^s - \sum_{\substack{t<m,u<n \\ \beta_{t,u}=c}} t^r u^s = \sum_{t<m,u<n} \delta_{t,u}^a t^r u^s = \sum_{t<m,u<n} \delta_{t,u}^a \sum_{i<p,j<q} \mu_{i,j} f_{i,j}^{m,n}$$

$$= \sum_{i<p,j<q} \mu_{i,j} \sum_{t<m,u<n} \delta_{t,u}^a f_{i,j}^{m,n,p,q}(t,u) = 0$$

Balanced morphisms

h is *balanced* if

$$h(a) \text{ and } h(b) \text{ contain the same number of } c, \text{ for all } a, b, c$$

Permutation array morphisms

Definition

A $\langle p, q\rangle$-morphism $h : [\![m]\!]^{**} \to [\![m]\!]^{**}$, is a permutation morphism, if

1. there exist a permutation π_1 of the set $[\![p]\!]$ and a permutation π_2 of the set $[\![q]\!]$ such that, for $i \in [\![m]\!]$ and $r \in [\![p]\!], s \in [\![q]\!]$,

$$h\left(0\right)_{\pi_1(r),\pi_2(s)} = \left(h\left(0\right)_{r,s} + 1\right) \bmod m,$$

2. for $i \in \{0, \ldots, m-1\}$

$$h\left(i\right) = \left(h\left(0\right) + i\right) \bmod m.$$

Permutation array morphisms

Lemma

Let $\alpha, \beta \in \llbracket m \rrbracket^{t,u}$ *such that* $S_{\alpha}^{(k)} = S_{\beta}^{(k)}$ *and* $(\beta = \alpha + z) \bmod m$. *Let* $h : \llbracket m \rrbracket^{**} \to \llbracket m \rrbracket^{**}$ *be a permutation morphism. Then* $S_{h(\alpha)}^{(k+1)} = S_{h(\beta)}^{(k+1)}$ *and* $h(\beta) = (h(\alpha) + z) \bmod m$.

Theorem

Let $h_0, \ldots, h_{k-1}$ *be a sequence of permutation morphisms,* $h_i : \llbracket m \rrbracket^{**} \to \llbracket m \rrbracket^{**}$. *Then*

$$S^{k}_{h_{k-1}(h_{k-2}(\cdots h_0(0)))} = S^{k}_{h_{k-1}(h_{k-2}(\cdots h_0(z)))}$$

for each $z \in \llbracket m \rrbracket$.

Theorem

Let $h_0, \dots, h_{k-1}$ be a sequence of permutation morphisms, where $h_i : [\![m]\!]^{**} \to [\![m]\!]^{**}$. Then $\{E_z\}_{z=1}^{m}$, where E_z consists of all positions in the word $h_{k-1}\left(h_{k-2}\left(\cdots h_0\left(z\right)\right)\right)$ containing the symbol 0, is a solution to the problem $PTE^2_{m,k}$.

Permutation array morphisms

Example

$h_0(0)=$ 2 1 0 $h_0(1)=$ 0 2 1 $h_0(2)=$ 1 0 2

$$h_1(0)=\begin{matrix}1 & 0\\ 0 & 2\\ 2 & 1\end{matrix}\qquad h_1(1)=\begin{matrix}2 & 1\\ 1 & 0\\ 0 & 2\end{matrix}\qquad h_1(2)=\begin{matrix}0 & 2\\ 2 & 1\\ 1 & 0\end{matrix}\qquad h_2=h_0$$

$h_2(h_1(h_0(0)))=$

2	1	0	1	0	2	1	0	2	0	2	1	0	2	1	2	1	0
1	0	2	0	2	1	0	2	1	2	1	0	2	1	0	1	0	2
0	2	1	2	1	0	2	1	0	1	0	2	1	0	2	0	2	1

Permutation array morphisms

$$A_0 = \{ \langle 0,2\rangle, \langle 0,4\rangle, \langle 0,7\rangle, \langle 0,9\rangle, \langle 0,12\rangle, \langle 0,17\rangle, \langle 1,1\rangle, \langle 1,3\rangle, \langle 1,6\rangle, \langle 1,11\rangle, \langle 1,14\rangle, \langle 1,16\rangle, \langle 2,0\rangle, \langle 2,5\rangle, \langle 2,8\rangle, \langle 2,10\rangle, \langle 2,13\rangle, \langle 2,15\rangle,$$

$$A_1 = \{ \langle 0,1\rangle, \langle 0,3\rangle, \langle 0,6\rangle, \langle 0,11\rangle, \langle 0,14\rangle, \langle 0,16\rangle, \langle 1,0\rangle, \langle 1,5\rangle, \langle 1,8\rangle, \langle 1,10\rangle, \langle 1,13\rangle, \langle 1,15\rangle, \langle 2,2\rangle, \langle 2,4\rangle, \langle 2,7\rangle, \langle 2,9\rangle, \langle 2,12\rangle, \langle 2,17\rangle \}$$

$$A_2 = \{ \langle 0,0\rangle, \langle 0,5\rangle, \langle 0,8\rangle, \langle 0,10\rangle, \langle 0,13\rangle, \langle 0,15\rangle, \langle 1,2\rangle, \langle 1,4\rangle, \langle 1,7\rangle, \langle 1,9\rangle, \langle 1,12\rangle, \langle 1,17\rangle, \langle 2,1\rangle, \langle 2,3\rangle, \langle 2,6\rangle, \langle 2,11\rangle, \langle 2,14\rangle, \langle 2,16\rangle \}$$

$$\sum_{\langle a_0,a_1\rangle \in A_i} a_0^0 a_1^0 = 18 \qquad \sum_{\langle a_0,a_1\rangle \in A_i} a_0^0 a_1^1 = 153$$

$$\sum_{\langle a_0,a_1\rangle \in A_i} a_0^1 a_1^0 = 18 \qquad \sum_{\langle a_0,a_1\rangle \in A_i} a_0^1 a_1^1 = 153$$

Future work: Composition of balanced morphisms

Let $H = (h_0, h_1, \ldots, h_{t-1})$ be a sequence of morphisms , $t \geq 1$,
where $h_i : [\![n_i]\!]^{**} \to [\![n_{i+1}]\!]^{**}$ is a $(m_{i,0}, m_{i,1})$- morphism,
$n_i, m_{i,0}, m_{i,1}, n_t \geq 2$, $i \in [\![t]\!]$, $n_0 = n$.
Let k of these morphisms $(1 \leq k \leq t)$ be balanced.
For $z \in [\![n]\!]$, let $H(z) = h_{t-1}(h_{t-2}(\ldots(h_0(z))))$.
Let B_z denote the set of all positions in $H(z)$ where the symbol 0 occurs.
Let $E = (E_0, \ldots, E_{n-1})$ where E_z is the set of all mixed-radix
$\langle\langle m_0, n_0\rangle, \ldots, \langle m_{t-1}, n_{t-1}\rangle\rangle$ notations of the pairs from B_z.

Theorem

The sequence E is a solution to the problem $PTE^d_{n,k}$.


Thank you