Picture Codes with finite deciphering delay

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Codes in the 1d world

- **1d**: Uniquely decipherable codes of strings

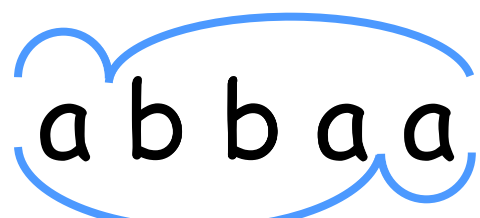
⇓

well-established theory

> $S \subseteq \Sigma^*$ is a **code** iff any $w \in \Sigma^*$ has at most one decomposition with strings of S

Examples: $S = \{a, abb\}$ is a code

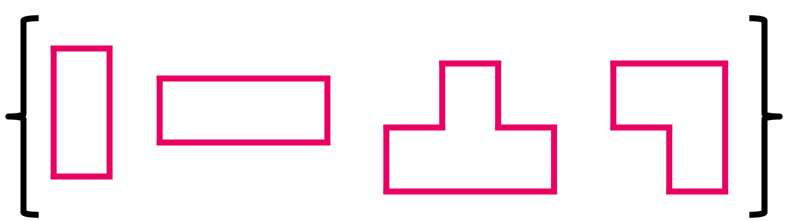
$S_1 = \{a, abba, bbaa\}$ is not a code

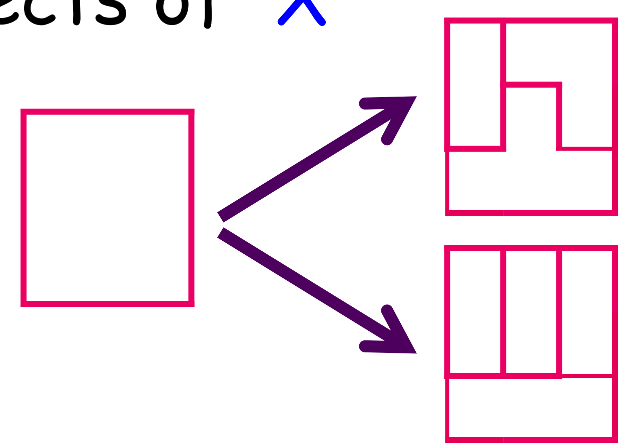


Codes in the 2d world

- **2d**: The notion of **code** is related to the problem of decomposing\tiling:

A set X of two-dimensional objects is a **code** iff any two-dimensional object has at most one decomposition\tiling with objects of X

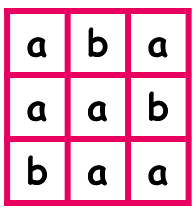
X = 



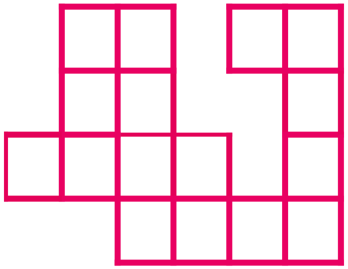
X is not a code.

Codes in the 2d world (ctd.)

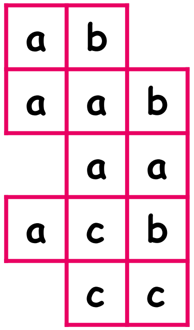
- 2d: Different two-dimensional objects



a picture



a polyomino

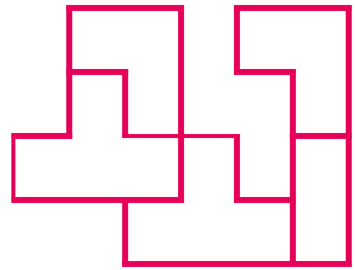


a labeled polyomino

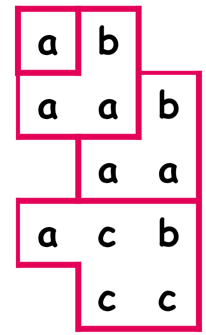
Different attempts to generalize the notion of **code**

Two-dimensional codes: previous works

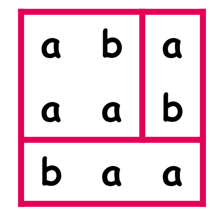
- Beauquier&Nivat (2003)



- M.Moczurad&W.Moczurad (2004)



- Bozapalidis&Grammaticopoulou (2006)



Unfortunately: always **undecidability** results already for finite sets

Two-dimensional codes of pictures: our definition

Let Σ^{**} be the set of all (rectangular) pictures over the alphabet Σ

Code is a set of pictures in Σ^{**} that tile in at most one way any picture in Σ^{**} [DLT2013]

Not surprisingly: undecidable!

Do we stop here? Nooooo!!!

Looking for decidable (sub-)families of codes, we extended 1d prefix codes and codes with finite deciphering delay

Our contributes

- **2d prefix** codes [CAI2013, DLT2013]: definition (prefix, strong prefix), decidability, polynomial parsing, maximality ...
- **2d** codes with **finite deciphering delay** [LATA2014]

Main results on finite deciphering delay

- Definition of sets of pictures with finite deciphering delay (d.d.) (the hardest part!!!)
- There exist codes with infinite d.d.
- Hierarchy of codes with finite d.d.

- Sets of pictures with finite deciphering delay are **codes**
- Decidability of finite codes with finite d.d.
- Polynomial algorithm for providing the unique decomposition of a picture in a code with finite d.d. (if any)

Let's start!

First: the two-dimensional objects

- **Pictures**: rectangular arrays of symbols taken from a finite alphabet

  Σ the finite alphabet

  Σ^{**} the set of all pictures over Σ

Second: the operation

- **Tiling star**

Tiling star of X

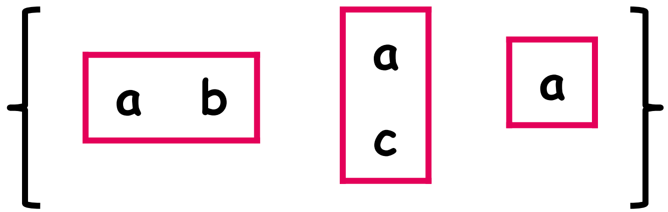
- In 1d

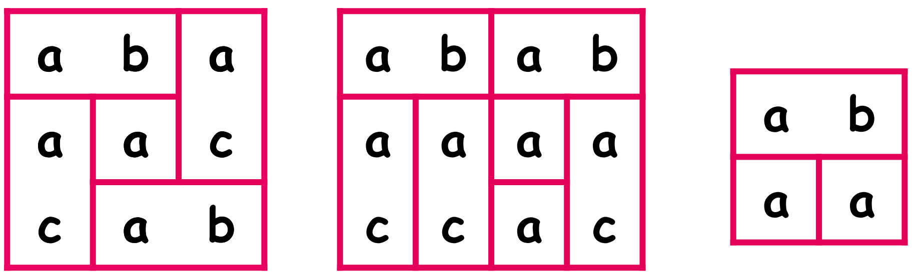
The **Kleene star** of a set **S**, denoted **S***, is the set of strings obtained by concatenating strings of **S**

- In 2d

Definition [Simplot '91]: The **tiling star** of X, denoted X**, is the set of pictures obtained by composing pictures of X in a way to cover a rectangular area.

Tiling star of X (ctd.)

Example: Let X =




$\in X^{**}$

Definition: If a picture $p \in X^{**}$ then p is **tilable** in X and the way to obtain p by composing pictures of X is a **tiling decomposition** of p on X

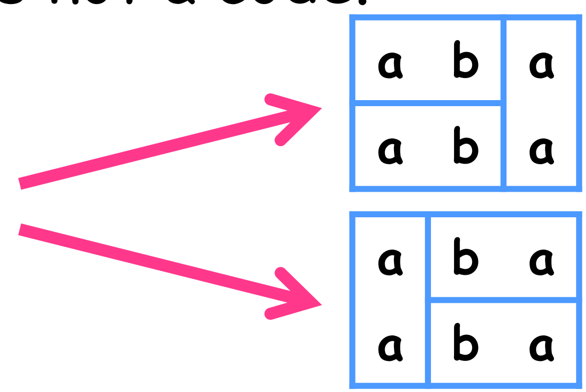
Two-dimensional code

Definition [DLT2013]: Let X be a set of pictures. X is a code iff any $p \in \Sigma^{**}$ has at most one tiling decomposition on X

Examples: Let $X = \left\{ \begin{array}{|cc|}\hline a & b \\ \hline\end{array} \quad \begin{array}{|c|}\hline a \\ b \\ \hline\end{array} \quad \begin{array}{|cc|}\hline a & a \\ a & a \\ \hline\end{array} \right\}$ X is a code.

Let $X_1 = \left\{ \begin{array}{|cc|}\hline a & b \\ \hline\end{array} \quad \begin{array}{|cc|}\hline b & a \\ \hline\end{array} \quad \begin{array}{|c|}\hline a \\ a \\ \hline\end{array} \right\}$ X_1 is not a code.

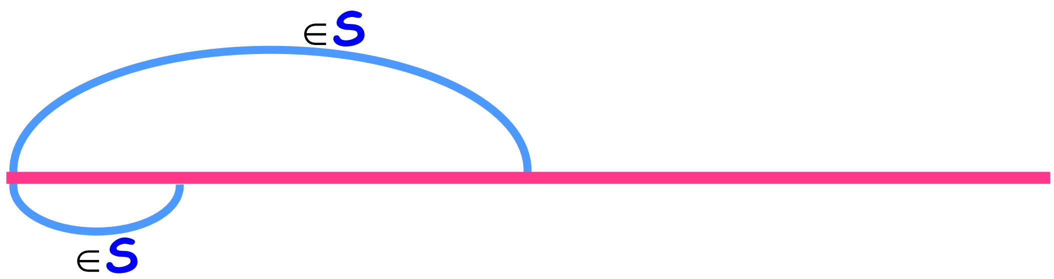
Consider $p = \begin{array}{|ccc|}\hline a & b & a \\ a & b & a \\ \hline\end{array}$



Proposition [DLT2013]: It is undecidable whether $X \subseteq \Sigma^{**}$ is a code

1d codes: prefix codes

A set of strings **S** is **prefix** if **no** string in **S** is prefix of another string in **S**



is not allowed

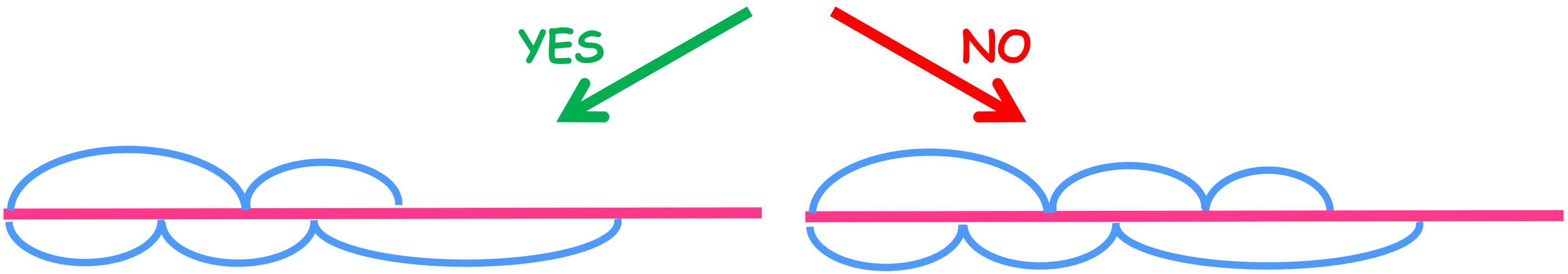
Fact: **S** prefix implies **S** code.

- The decoding of a given coded-message can start **immediately**: reading the string from left-to-right, only **one** element in **S** matches the input

1d codes: codes with finite d.d.

- Codes with finite deciphering delay generalize prefix codes: the decoding of a given coded-message can start after reading a finite number of code words.

Example: **finite deciphering delay** equal to **2**



- Prefix codes have d.d. 0, and are **instantaneous** codes

- Both prefix and finite d.d. codes deal with the "initial part" of a decomposition

One-dimensional deciphering delay

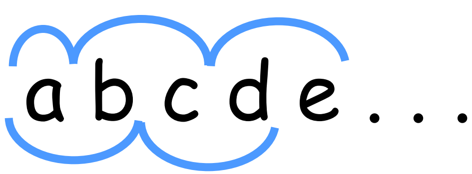
- 1d: Deciphering delay equal to **k** means that:

when a string has two prefixes in S^+, that start with **different strings** of **S**, then the shortest one has, at most, **k** strings.

A set of strings **S** has **finite deciphering delay** if there is an integer **k** such that for all **x**, **x'** in **S**, **y** in S^k, **y'** in S^*, **xy** prefix of **x'y'** implies **x=x'**

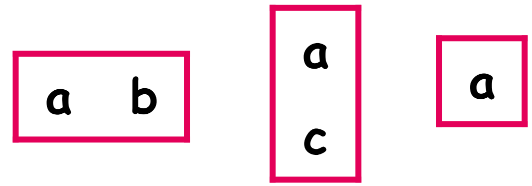
The **smallest** integer **k** satisfying the condition is the **deciphering delay** of **S**.

Example: The set $S = \{a, ab, bc, cd, de\}$ has d.d. 2

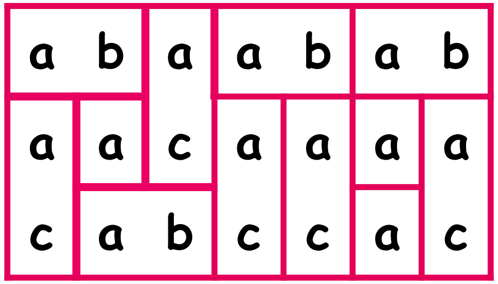


Partial decompositions

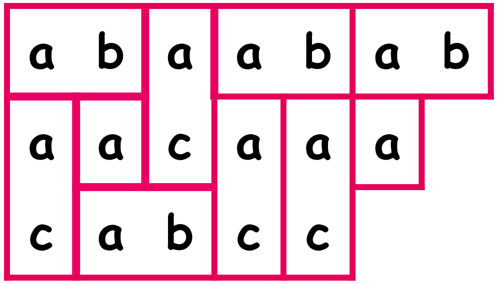
Example: Let X = {



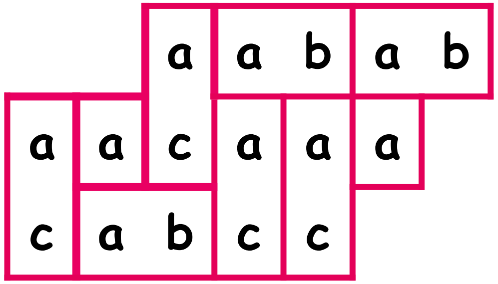
}



Tiling decomposition

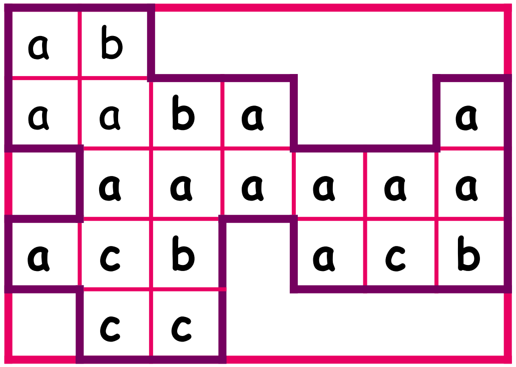


Initial part



Intermediate step

Dealing with polyominoes

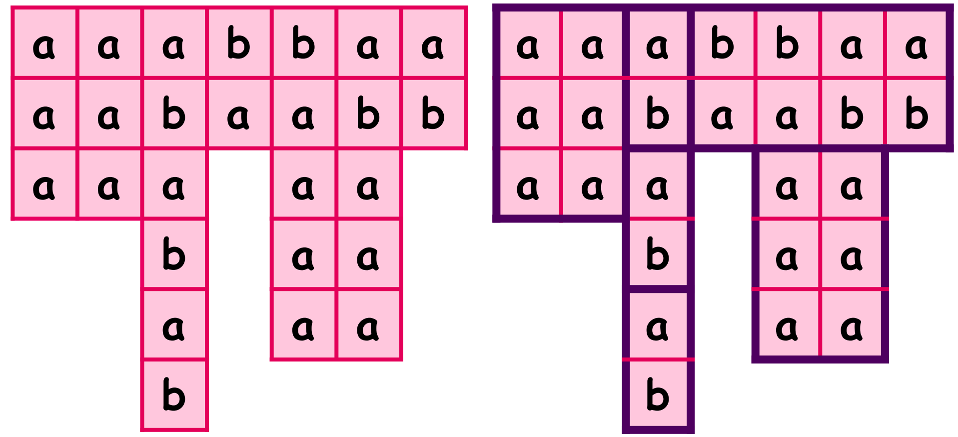


(labeled) polyomino

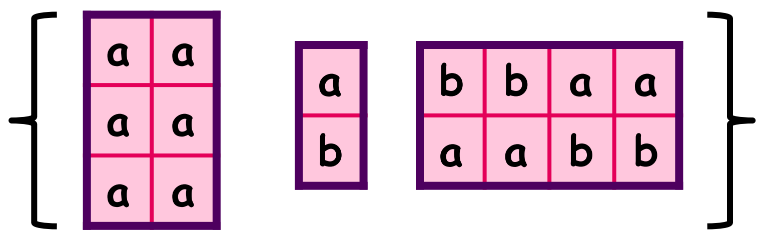
- To use consistent notation dealing with pictures and polyominoes, we consider a polyomino as plugged in its **minimal bounding box**
- The **domain** of a polyomino c, denoted **dom**(c), is the set of all positions (i,j) occupied inside its minimal bounding box
- Position (1,1) is assigned to the top-left corner of the minimal bounding box

Polyomino tilable in a set of pictures

Consider the polyomino c =



and the set X =



c can be obtained by composing pictures in X.

We say that c is tilable in X

Toward the definition: idea

- 2d: Deciphering delay equal to k means that, if there is a polyomino with two "prefixes" tilable in X, such that the two "prefixes" start with different pictures of X, then the smallest one has, at most, k code pictures.

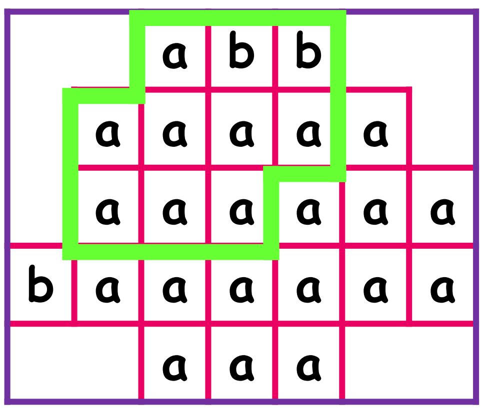
We fix the top-left to bottom-right scanning direction

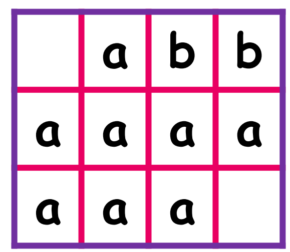
Toward the definition: difficulties

- **Prefix** in the framework of polyominoes
- **First picture** of a polyomino decomposition
- **Extension** of a polyomino decomposition from **k** to more than **k** pictures

Prefix of polyominoes: examples

Example: Consider the polyominoes

x = 

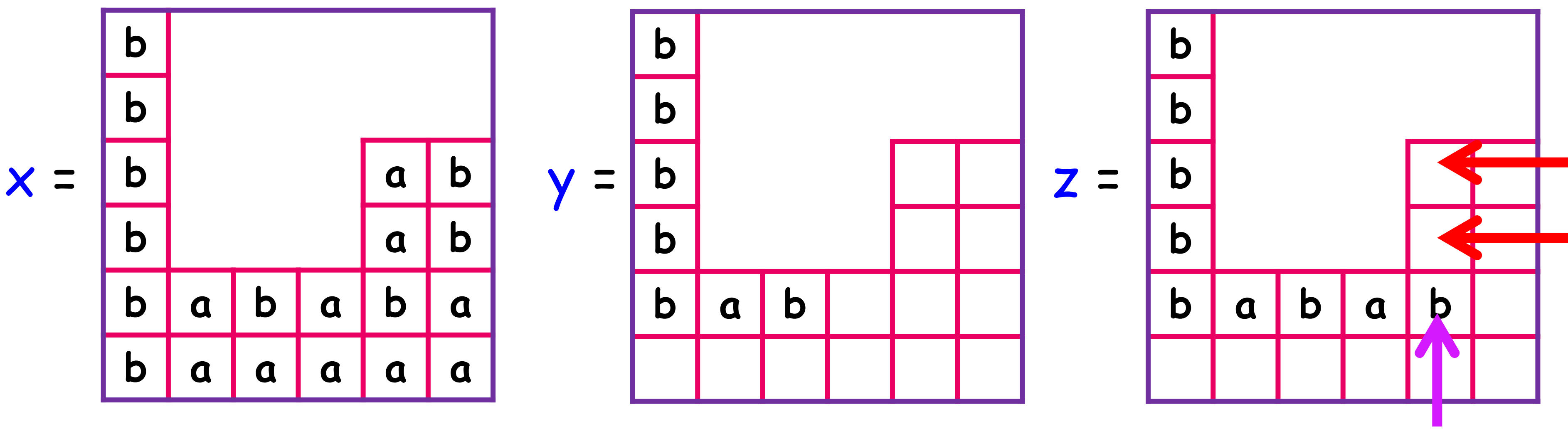
and y = 

We have that y is a prefix of x

but... be careful....

Prefix of polyominoes: examples

Recall: we fixed the top-left to bottom-right scanning direction and we impose a sort of "**row and column prefixness**":
If **c** is prefix of **c'**, each row\column of **c** must be prefix of the corresponding row\column in **c'**



We have that y is prefix of x whereas z is not prefix of x

The position (5, 5) is in dom(z) but the positions (3, 5) and (4, 5) are in dom(x) \ dom(z).

Prefix of polyominoes: definition

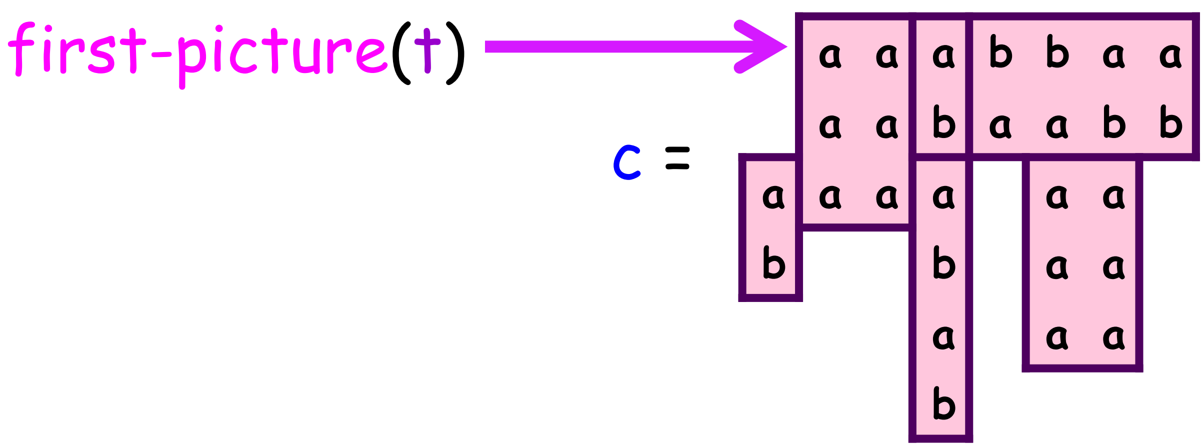
Definition: Let c and c' polyominoes. We say that c is prefix of c', and we write $c \lhd c'$, if the following conditions are verified:

- $dom(c) \subseteq dom(c')$
- the labels of the common positions of c and c' are equal
- if the position $(i_0, j_0) \in dom(c)$ then, for all $(i, j) \in dom(c')$ with $i \leq i_0$ and $j \leq j_0$, we have that $(i, j) \in dom(c)$

Remark: the third condition imposes some "row and column prefixness"

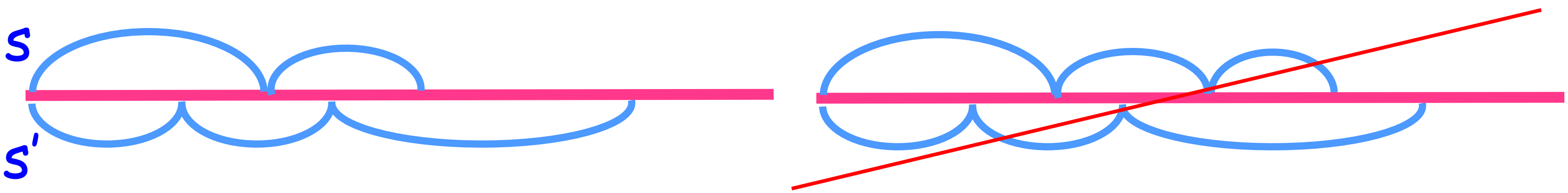
First picture of a decomposition

Definition: Let $X \subseteq \Sigma^{**}$, let c be a polyomino tilable in X and t a tiling decomposition of c in X. The picture in t that contains the top-left corner of c is the **first-picture(t)**.



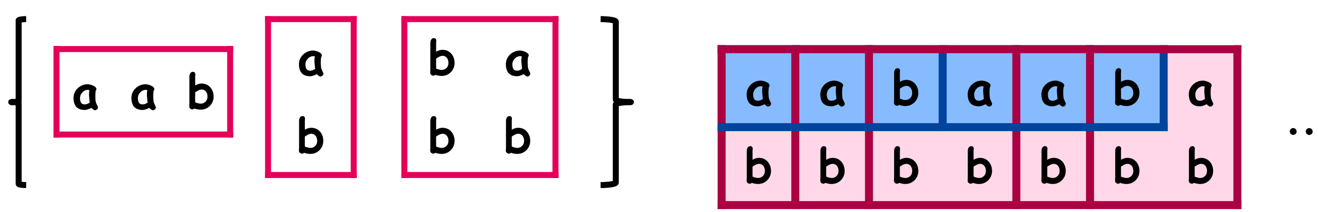
Extension of a decomposition: idea

1d: for any s, s' in S* with the first string in s different from the first string in s', s prefix of s', s with k code strings, it is not possible to "extend" s consistently with s'



2d: we need to consider an extension step that enlarges the prefix all along its "frontier" (not only in one direction)

Consider X =



Extension of a decomposition: definition

Definition: Let $X \subseteq \Sigma^{**}$, c, c', c'' polyominoes tilable in X, with $c \lhd c'$, t, t'' tiling decompositions of c and c'' in X.

We say that the decomposition t'' **extends** t consistently with c' if:

- codes pictures that occur in t, occur in t'' in the same positions
- labels of the common positions of c' and c'' are the same
- dom(c'') contains all positions $(i,j) \in$ dom(c')$\backslash \in$ dom(c) such that $(i,j-1)$ or $(i-1,j-1)$ or $(i-1,j)$ is in dom(c)

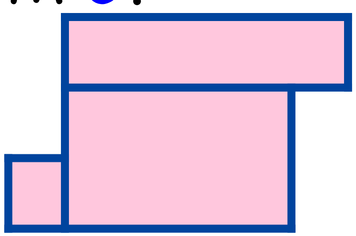
Definition of deciphering delay

Definition: Let $X \subseteq \Sigma^{**}$.

X has **finite deciphering delay** if it is prefix or there is a $k>0$ such that:

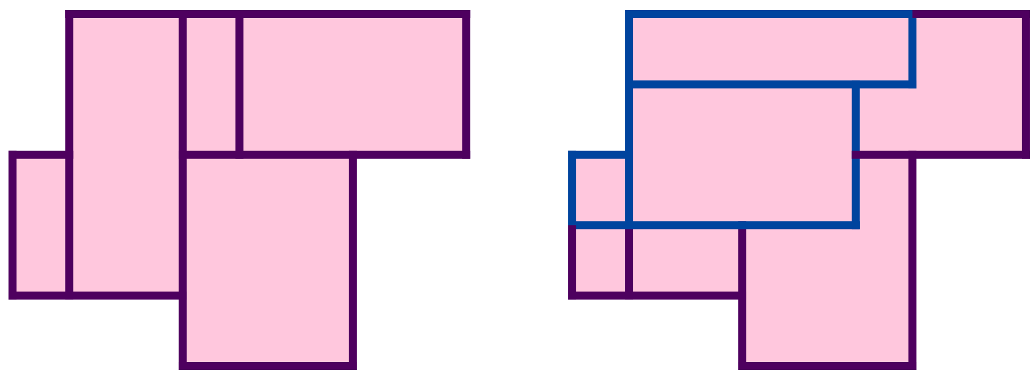
for any c, c' polyominoes tilable in X, with tiling decompositions t, t' and $c \lhd c'$, t with k code pictures, first-picture(t) $\neq$ first-picture(t'), there is no c'' with a tiling decomposition t'' in X ($t'' \neq t$) that extends t consistently with c'.

c =



t

c′ =



t′

Definition of deciphering delay (ctd.)

Remark: If the condition of the definition holds for some integer k, then it holds for all $k' \geq k$. The smallest integer k satisfying the condition is the deciphering delay of X

Remark: Prefix codes can be considered languages with deciphering delay $k = 0$

An example

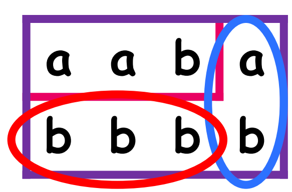
Let $X = \left\{ \boxed{a\ a\ b} \quad \boxed{\begin{array}{c} a \\ b \end{array}} \quad \boxed{\begin{array}{cc} b & a \\ b & b \end{array}} \right\}$ X has deciphering delay 1.

Two cases: we consider only the first one (the second one is analogous)

$c = \boxed{a\ a\ b}$ prefix of $c' = \boxed{\begin{array}{cccc} a & a & b & a \\ b & b & b & b \end{array}}$ with tiling decompositions

$t = \boxed{a\ a\ b}$ and $t' = \boxed{\begin{array}{c|c|cc} a & a & b & a \\ b & b & b & b \end{array}}$ first-picture(t) $\neq$ first-picture(t')

There is no c'' polyomino tilable in X with a tiling decomposition t'' ($t'' \neq t$) that extends t consistently with c'.



Main results (again)

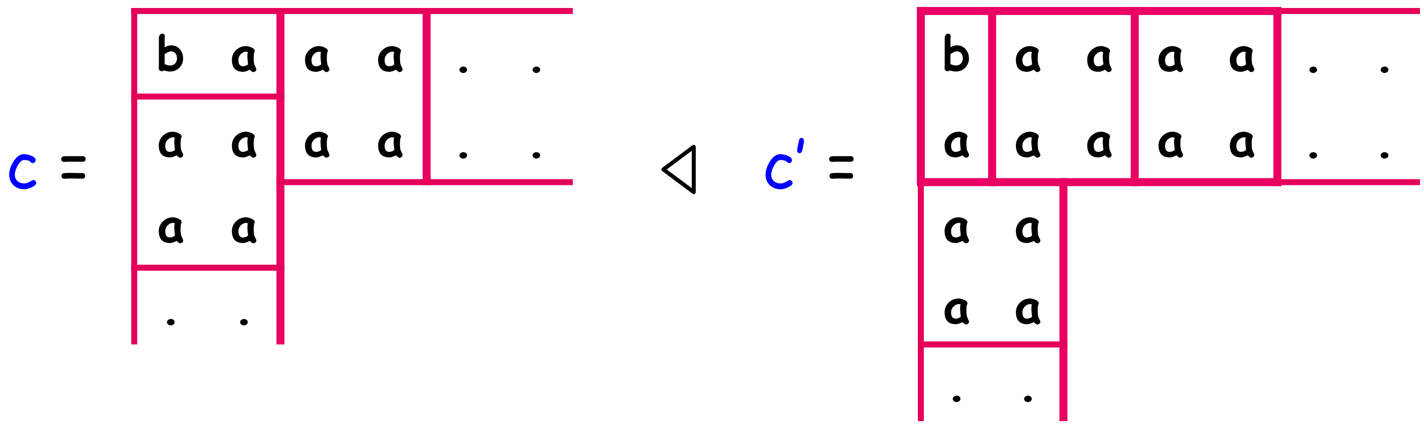
- Definition of sets of pictures with finite deciphering delay (d.d.) (the hardest part!!!)
- There exist codes with infinite d.d.
- Hierarchy of codes with finite d.d.

- Sets of pictures with finite deciphering delay are codes
- Decidability of finite codes with finite d.d.
- Polynomial algorithm for providing the unique decomposition of a picture in a code with finite d.d. (if any)

Infinite deciphering delay

Example: Let $X = \left\{ \begin{bmatrix} b & a \end{bmatrix}, \begin{bmatrix} b \\ a \end{bmatrix}, \begin{bmatrix} a & a \\ a & a \end{bmatrix} \right\}$ X is a code.

X has an **infinite** deciphering delay. Consider



with tiling decompositions t and t′, as in the figure, and first-picture(t) ≠ first-picture(t′).

There is always c″ polyomino tilable in X with a tiling decomposition t″ (t″ ≠ t) that extends t consistently with c′.

Hierarchy

Theorem: For any **k** there is a set **X** that has **deciphering delay k**.

Proof:

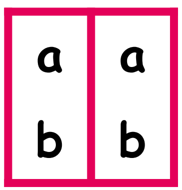
Consider the set $X_k = \left\{ \begin{bmatrix} a \\ b \end{bmatrix}, \overbrace{\begin{bmatrix} a & .. & a \end{bmatrix}}^{k\ \text{times}} \begin{bmatrix} b \end{bmatrix}, \overbrace{\begin{bmatrix} b & .. & b \\ b & .. & b \end{bmatrix}}^{k\ \text{times}} \begin{bmatrix} a \\ b \end{bmatrix} \right\}$

For any **k**, X_k has **deciphering delay k**.

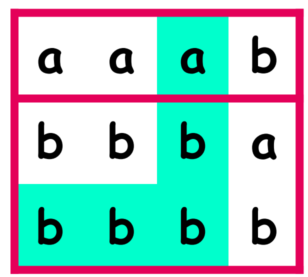
Let's prove for **k**=3.

Hierarchy (ctd.)

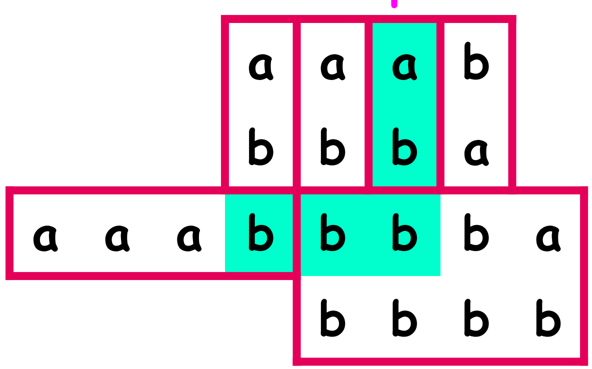
$X_3 = \left\{ \begin{bmatrix} a \\ b \end{bmatrix} \; \begin{bmatrix} a & a & a & b \end{bmatrix} \; \begin{bmatrix} b & b & b & a \\ b & b & b & b \end{bmatrix} \right\}$ has deciphering delay ≥ 2.

$c =$ 

$\lhd$

$c' =$ 

with tiling decompositions t and t', as in the figure, t with 2 code pictures and first-picture(t) $\neq$ first-picture(t').

There is $c'' =$ 

polyomino tilable in X with a tiling decomposition t'' that extends t consistently with c'.

Hierarchy (ctd.)

X_3 has deciphering delay = 3. Consider

c =

a	a	a
b	b	b

$\triangleleft$

c' =

a	a	a	b
b	b	b	a
b	b	b	b

with tiling decompositions t and t', as in the figure, t with 3 code pictures and first-picture(t) $\neq$ first-picture(t').

There is no c'' polyomino tilable in X with a tiling decomposition t'' ($t'' \neq t$) that extends t consistently with c'.

Future works

- Consider **maximality** of codes with **finite deciphering delay**
- Consider **Bifix Codes** and other families of decidable picture codes
- Remove the **finiteness** hypothesis and consider **deterministically recognizable** decidable codes

Gracias