Weighted Automata and Logics for Infinite Nested Words

Manfred Droste and Stefan Dück

Leipzig University, Germany

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Nested Words (Alur and Madhusudan JACM 2009)

Usage

Representation of data with linear and hierarchical structure (e.g. structured programs, XML documents)

$\Sigma = \{a\ ,\ b\}$, $\hat{\Sigma} = \{a,\ \langle a\ ,\ a\rangle\ ,\ b\ ,\ \langle b\ ,\ b\rangle\}$

- word over Σ : $a \to b \to a \to b \to a \to b \to a$

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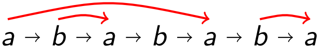

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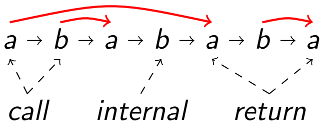

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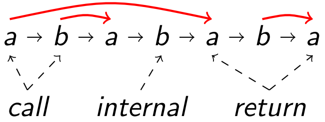

- representation over $\hat{\Sigma}$: $\langle a\ \langle b\ a\rangle\ b\ a\rangle\ \langle b\ a\rangle$

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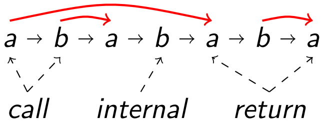

- representation over $\hat{\Sigma}$: $\langle a\ \langle b\ a\rangle\ b\ a\rangle\ \langle b\ a\rangle$
- with *nesting relation* ν : $(w, \nu) \in NW(\Sigma)$

$= (ababab a, \{(1,5),(2,3),(6,7)\})$

Nested Word Automata

NWA (Alur and Madhusudan)

$\mathcal{A} = (Q, q_0, (\delta_{\mathsf{call}}, \delta_{\mathrm{int}}, \delta_{\mathsf{ret}}), Q_{\mathrm{f}})$

$\delta_{\mathsf{call}}, \delta_{\mathrm{int}} : Q \times \Sigma \rightarrow Q$

$\delta_{\mathsf{ret}} : Q \times {\color{red}Q} \times \Sigma \rightarrow Q$

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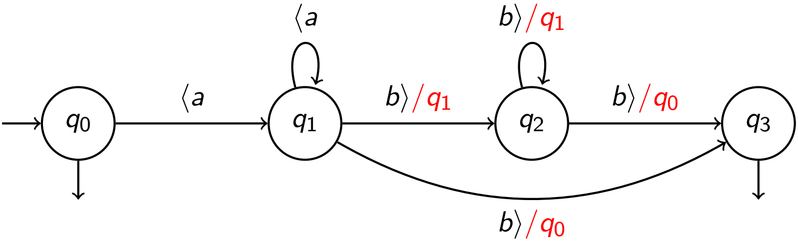

$\Rightarrow L(\mathcal{A}) = \{\ (\langle a)^k\ (b\rangle)^k \mid k \geq 0\}$

MSO Logic for Nested Words

Definition ($MSO(NW(\Sigma))$)

$$\beta ::= \mathrm{Lab}_a(x) \mid \mathrm{call}(x) \mid \mathrm{ret}(x) \mid \nu(x,y) \mid x \leq y \mid x \in X \mid \neg\beta \mid \beta \wedge \beta \mid \forall x.\beta \mid \forall X.\beta$$

with $a \in \Sigma$, $x, y, X \in \mathcal{V}$, $\mathcal{V}$ finite set of FO- and SO-variables

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Theorem (Alur and Madhusudan)

L language of nested [ω-] words over Σ, TFAE:

(1) *$L = L(\mathcal{A})$ for an [sM] NWA $\mathcal{A}$*

(2) *$L = L(\beta)$ for an $MSO(NW(\Sigma))$-sentence β*

Our goal: Quantitative version of this result

Valuation Monoids (Droste and Meinecke 2012)

Definition

ω-valuation monoid $(D, +, \mathrm{Val}^\omega, 0)$:

- complete monoid $(D, +, 0)$ (infinite sums defined)
- *ω-valuation function* $\mathrm{Val}^\omega : D^\omega \to D$
  $\mathrm{Val}^\omega((d_i)_{i \in \mathbb{N}}) = 0$ if $d_i = 0$ for an $i \in \mathbb{N}$

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Examples

1. totally complete semirings $(\mathbb{K}, +, \cdot, 0, 1)$
2. Chatterjee, Doyen, Henzinger 2008:

$$(\mathbb{R} \cup \{-\infty, \infty\}, \sup, \lim\ \mathrm{avg}, -\infty),$$

$$\lim\ \mathrm{avg}((d_i)_{i\in\mathbb{N}}) := \liminf_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} d_i$$

Weighted Stair Muller NWA

wsMNWA

$$\mathcal{A} = (Q, I, (\delta_{call}, \delta_{\text{int}}, \delta_{\text{ret}}), \mathfrak{F}),\ \mathfrak{F} \subseteq 2^Q$$

$$\delta_{\text{call}}, \delta_{\text{int}} : Q \times \Sigma \times Q \to D$$

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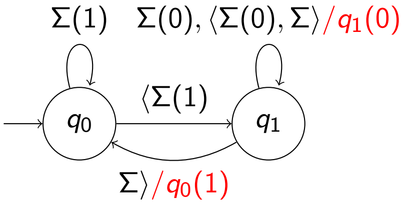
$\mathcal{A}$ over $D = (\bar{\mathbb{R}}, \sup, \lim \text{avg}, -\infty)$ with $\mathfrak{F} = \{\{q_0\}\}$



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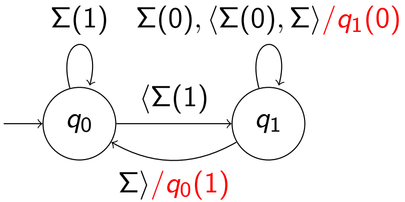

$$\|\mathcal{A}\|(nw) := \sum_{r \text{ acc}} \text{Val}^{\omega}((wt_{\mathcal{A}}(r, nw, i))_{i \in \mathbb{N}})$$

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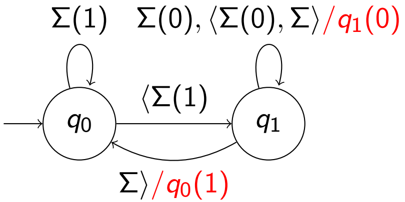

$\|\mathcal{A}\|(nw)$

$:= \sum_{r \text{ acc}} \text{Val}^{\omega}((wt_{\mathcal{A}}(r, nw, i))_{i \in \mathbb{N}})$

$= \lim\text{ avg}((wt_{\mathcal{A}}(r, nw, i))_{i \in \mathbb{N}})$

= 'ratio' of top-level positions of nw

Weighted MSO Logic – based on Droste and Gastin 2006

Definition ($MSO(D, NW(\Sigma))$)

$$\varphi ::= d \mid \beta \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \forall x.\varphi \mid \exists x.\varphi \mid \exists X.\varphi$$

with $d \in D$, $\beta \in MSO(NW(\Sigma))$ (*boolean*), $x, y, X \in \mathcal{V}$ as before

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Definition (ω-pv-monoid – Droste and Meinecke)

product ω-valuation monoid $(D, +, \mathrm{Val}^\omega, \diamond, 0, 1)$

- ω-valuation monoid $(D, +, \mathrm{Val}^\omega, 0)$
- $1 \in D$, $\mathrm{Val}^\omega(1^\omega) = 1$
- $\diamond : D^2 \to D$,
  $0 \diamond d = d \diamond 0 = 0$ and $1 \diamond d = d \diamond 1 = d$ $\forall d \in D$

Semantics (1)

given

- $\varphi \in MSO(D, NW(\Sigma))$
- $nw \in NW^{\omega}(\Sigma)$
- σ assignment of free variables of φ to positions / set of positions

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define $[\![\varphi]\!](nw, \sigma) \in D$ inductively as follows

Semantics (2)

$$[\![\, \beta \,]\!](nw, \sigma) := \mathbb{1}_{L(\beta)}(nw, \sigma)$$

$$[\![\, d \,]\!](nw, \sigma) := d \quad \text{for all } d \in D$$

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$$\llbracket \beta \rrbracket(nw, \sigma) := \mathbb{1}_{L(\beta)}(nw, \sigma)$$
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$$\llbracket \varphi \vee \psi \rrbracket(nw, \sigma) := \llbracket \varphi \rrbracket(nw, \sigma) + \llbracket \psi \rrbracket(nw, \sigma)$$
$$\llbracket \varphi \wedge \psi \rrbracket(nw, \sigma) := \llbracket \varphi \rrbracket(nw, \sigma) \diamond \llbracket \psi \rrbracket(nw, \sigma)$$

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$$[\![\exists x.\varphi]\!](nw, \sigma) := \sum_{i \in \mathbb{N}} ([\![\varphi]\!](nw, \sigma[x \to i]))$$

$$[\![\exists X.\varphi]\!](nw, \sigma) := \sum_{I \subseteq \mathbb{N}} ([\![\varphi]\!](nw, \sigma[X \to I]))$$

$$[\![\forall x.\varphi]\!](nw, \sigma) := \mathrm{Val}^{\omega}([\![\varphi]\!](nw, \sigma[x \to i]))_{i \in \mathbb{N}}$$

Logic Fragments

Motivation: $REC \subsetneq wMSO$ for finite words (Droste and Gastin)

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with $d \in D$, $\beta \in MSO(NW(\Sigma))$

Definition (φ – syntactically restricted)

- *$\forall$-restricted*: subformula $\forall x.\psi \Rightarrow \psi$ almost boolean
- *strongly-$\wedge$-restricted*: subformula $\psi \wedge \theta$
  $\Rightarrow \psi$ and θ almost boolean or ψ or θ boolean

First Main Result

Theorem

D regular ω-pv-monoid (constant series are recognizable),
$S : NW^\omega(\Sigma) \to D$, TFAE:

(1) $S = \|\mathcal{A}\|$ *for a wsMNWA* $\mathcal{A}$

(2) $S = [\![\varphi]\!]$ *for a syntactically restricted sentence* φ

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Additional assumptions on $D \Rightarrow$ less restricted formulas possible

Logic Fragments (2)

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We call $\varphi \in MSO(D, NW(\Sigma))$

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2. *$\wedge$-restricted*: subformula $\psi \wedge \theta$
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3. *weakly-$\wedge$-restricted*: subformula $\psi \wedge \theta$
   $\Rightarrow$ ψ almost boolean or const(ψ) and const(θ) commute

(const(φ) := set of all elements of D occurring in φ)

Second Main Result

Theorem

1.
   - *D left-distributive*
   - *φ $\wedge$-restricted*

   $\Rightarrow$ *$\exists$ strongly-$\wedge$-restricted φ' : $[\![\varphi']\!] = [\![\varphi]\!]$*
2.
   - *D cc-ω-valuation semiring*
   - *φ weakly-$\wedge$-restricted*

   $\Rightarrow$ *$\exists$ strongly-$\wedge$-restricted φ' : $[\![\varphi']\!] = [\![\varphi]\!]$*
3. *If φ is here $\forall$-restricted, we can construct φ' also $\forall$-restricted.*

Restating First Main Result

Theorem

D ω-pv-monoid, $S : NW^\omega(\Sigma) \to D$

1. *D regular, TFAE:*
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Thank you for your attention!

Definition

ω-pv-monoid $(D, +, \mathrm{Val}^\omega, \diamond, 0, 1)$

- *associative/commutative*: $\diamond$ is associative/commutative
- *left-+-distributive*: $d \diamond \sum_{i \in I} d_i = \sum_{i \in I} (d \diamond d_i)$ (*right-+-distributive*, *+-distributive* resp.)
- *ω-valuation semiring*: +-distributive and associative
- *left-multiplicative*: $d \diamond Val^\omega((d_i)_{i \in \mathbb{N}}) = Val^\omega(d \diamond d_1, (d_i)_{i \geq 2})$
- *left-Val^ω-distributive*: $d \diamond Val^\omega((d_i)_{i \in \mathbb{N}}) = Val^\omega((d \diamond d_i)_{i \in \mathbb{N}})$
- *left-distributive*: left-+-distributive and (left-multiplicative or left-Val^ω-distributive)
- *conditionally commutative*: $(\forall j < i : d_i \diamond d'_j = d'_j \diamond d_i) \Rightarrow Val^\omega((d_i)_{i \in \mathbb{N}}) \diamond Val^\omega((d'_i)_{i \in \mathbb{N}}) = Val^\omega((d_i \diamond d'_i)_{i \in \mathbb{N}})$
- *cc-ω-valuation semiring*: semiring D conditionally commutative and left-distributive

App. Complete Monoid

Definition (complete monoid)

A monoid $(D, +, 0)$ is *complete* if it has infinitary sum operations $\sum_I : D^I \to D$ for any index set I such that

- $\sum_{i \in \emptyset} d_i = 0$
- $\sum_{i \in \{k\}} d_i = d_k$
- $\sum_{i \in \{j,k\}} d_i = d_j + d_k$ for $j \neq k$
- $\sum_{j \in J} (\sum_{i \in I_j} d_i) = \sum_{i \in I} d_i$
  if $\bigcup_{j \in J} I_j = I$ and $I_j \cap I_k = \emptyset$ for $j \neq k$

Note that in every complete monoid the operation $+$ is commutative.

App. wsMNWA - Acceptance

wsMNWA $\mathcal{A} = (Q, I, (\delta_{call}, \delta_{\mathrm{int}}, \delta_{\mathrm{ret}}), \mathfrak{F})$, $nw = (a_1 a_2 ..., \nu)$

Definition (top-level)

We call $i \in \mathbb{N}$ a *top-level position* if there exist no positions $j, k \in \mathbb{N}$ with $j < i < k$ and $\nu(j, k)$.

Definition (run)

run r over nw: $r = (q_0, q_1, ...)$

$Q^t_\infty(r) := \{q \in Q \mid q_i = q \text{ for infinitely many top-level positions } i\}$

r *accepting* $:\Leftrightarrow q_0 \in I$ and $Q^t_\infty(r) \in \mathfrak{F}$

App. wsMNWA - Weights

Definition (weights)

$$wt_{\mathcal{A}}(r, nw, i) := \begin{cases} \delta_{call}(q_{i-1}, a_i, q_i) & \text{, if } i \text{ is a call} \\ \delta_{int}(q_{i-1}, a_i, q_i) & \text{, if } i \text{ is an internal} \\ \delta_{ret}(q_{i-1}, q_{j-1}, a_i, q_i) & \text{, if } \nu(j, i) \end{cases}$$

Definition (behaviour of $\mathcal{A}$)

Behaviour of the automaton $\mathcal{A}$, $\|\mathcal{A}\| : NW^{\omega}(\Sigma) \to D$:

$$\|\mathcal{A}\|(nw) := \sum_{r \text{ acc}} \mathrm{Val}^{\omega}((wt_{\mathcal{A}}(r, nw, i))_{i \in \mathbb{N}})$$

App. Complexity – Alur and Madhusudan 2009

	Decision problems for automata		
	Emptiness	Universality	Inclusion
DFA	NLOGSPACE	NLOGSPACE	NLOGSPACE
NFA	NLOGSPACE	PSPACE	PSPACE
PDA	PTIME	Undecidable	Undecidable
NWA	PTIME	PTIME	PTIME
Nondet NWA	PTIME	EXPTIME	EXPTIME

App. Main Theorem - Components

Lemma (Closure under conjunction)

θ, ψ subformulas of a syntactically restricted formula φ:

$[\![\theta]\!]$ *and* $[\![\psi]\!]$ *regular* $\Rightarrow [\![\theta \wedge \psi]\!]$ *regular*

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Lemma (Closure under restricted $\forall x$)

φ almost boolean $\Rightarrow$ $[\![\forall x.\varphi]\!]$ regular

App. Main Theorem - Ideas

Series S *regular step function* :⇔ ∃ representation:

$$S(nw) = \sum_{i=1}^{k} d_i \mathbb{1}_{L_i}(nw)$$

- L_i: regular languages of nested ω-words
- (L_i) partition of $NW^\omega(\Sigma)$
- $d_i \in D$
- $S(nw) = d_i \Leftrightarrow nw \in L_i$

↷

- (regular) ω-valuation-monoid D:
  S regular step function ⇒ S regular
- regular step functions closed under $\diamond$ and $+$ (pointwise)
- φ almost boolean ⇒ $[\![\varphi]\!]$ regular step function

App. Definition Nesting Relation

- *nesting relation* ν
  - of length $\ell \geq 0$: subset of $\{-\infty, 1, ..., \ell\} \times \{1, ..., \ell, \infty\}$
  - over $\mathbb{N}$: subset of $(\{-\infty\} \cup \mathbb{N}) \times (\mathbb{N} \cup \{\infty\})$

(i) $\nu(i,j) \Rightarrow i < j$,

(ii) $\forall i : 1 \leq i \leq \ell \Rightarrow |\{j : \nu(i,j)\}| \leq 1 \wedge |\{j : \nu(j,i)\}| \leq 1$,
(ii') $\forall i : i \in \mathbb{N} \Rightarrow |\{j : \nu(i,j)\}| \leq 1 \wedge |\{j : \nu(j,i)\}| \leq 1$
(at most one nesting edge per position (except $-\infty, \infty$)),

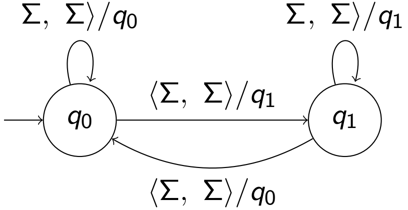
(iii) $\nu(i,j) \wedge \nu(i',j') \wedge i < i' \Rightarrow j < i' \vee j > j'$
(nesting edges do not cross),

(iv) $(-\infty, \infty) \notin \nu$.

App. sMNWA Example

$L := \{nw \in NW^{\omega}(\Sigma) \mid nw$ has only finitely many pending calls$\}$

sMNWA with $\mathfrak{F} = \{\{q_0\}, \{q_1\}\}$



App. Class VPL

$$\text{Regular} \subset \mathit{VPL} \subset \mathit{CFL}$$

$$\begin{aligned} \mathit{VPL} &= \text{visibly pushdown languages} \\ &\mathrel{\hat{=}} \text{regular languages of nested words} \end{aligned}$$

- Closed under $\cup$, $\cap$ and complement
- Substantially more expressive than regular languages
- Usage: Representation of data with linear and hierarchical structure (e.g. structured programs, XML documents)