On Sorting of Signed Permutations by Prefix and Suffix Reversals and Transpositions

AlCoB'2014, Tarragona, Spain

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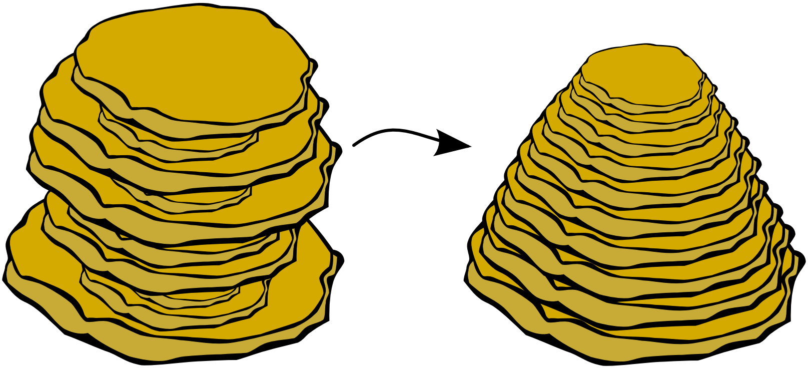
Partially supported by FAPESP (grants 2013/01172-0 and 2013/08293-7), CNPq (grants 477692/2012-5 and 483370/2013-4), and FAEPEX (process 396/2014)

July 3, 2014

Introduction

Dweighter [1], 1975: *The Pancake Flipping Problem*



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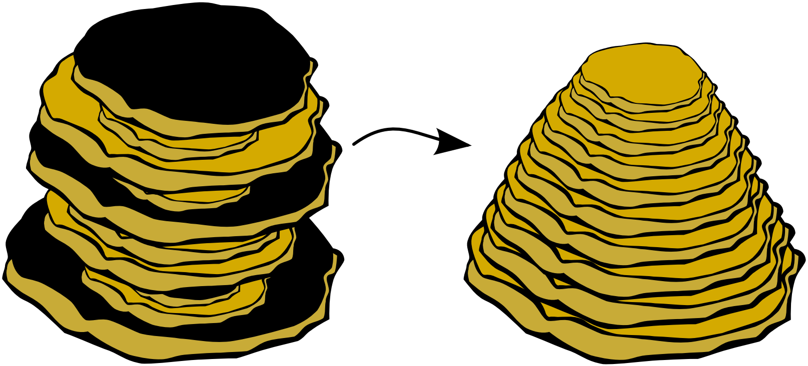
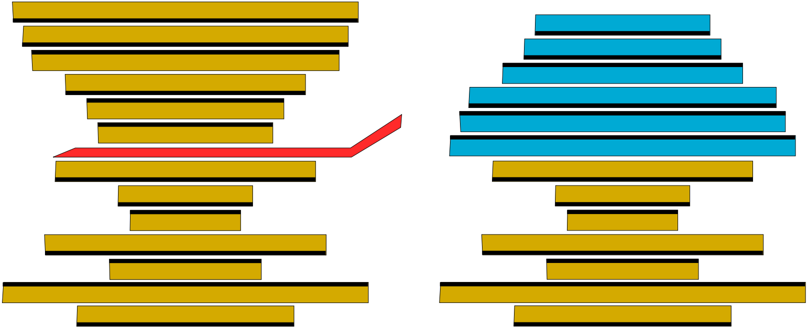
Gates and Papadimitriou [2], 1979: *The Burnt Pancake Flipping Problem*



Introduction

Allowed moves: prefix reversals



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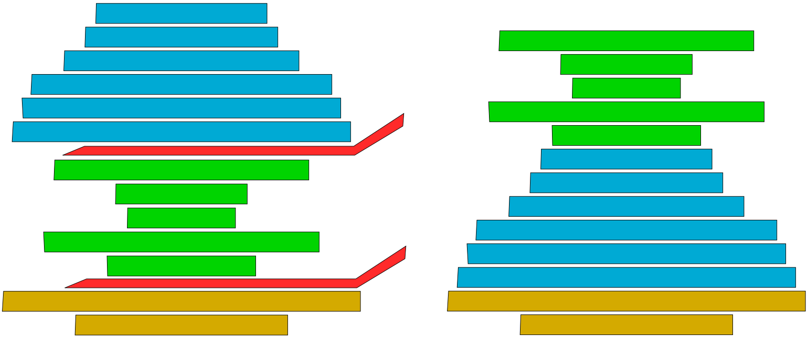
- Mutations allow evolutions and the difference between two genomes indicates their evolutionary distance
- Find scenarios that show how to transform one genome into another
    - minimum number of rearrangements that allow the transformation

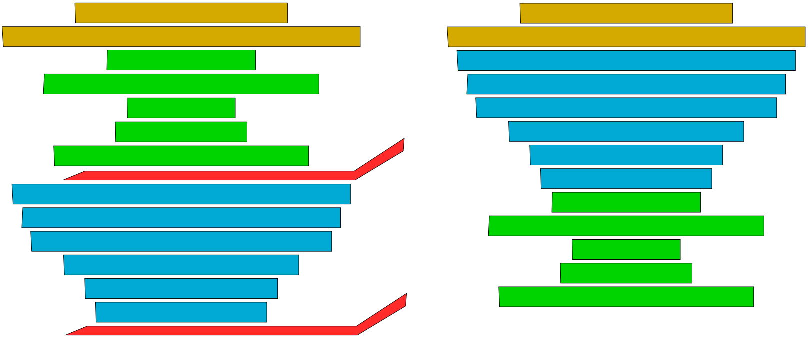
Introduction

Dias and Meidanis [3], 2002: Prefix Transpositions
Sharmin *et al.* [4], 2010: *Pancake Flipping with Two Spatulas*



Introduction

Lintzmayer and Dias [5], 2014: Suffix operations



Definitions

- Permutation:
  $\pi = (\pi_0 = +0 \ \pi_1 \ \pi_2 \ \ldots \ \pi_n \ \pi_{n+1} = +(n+1))$
  where $|\pi_i| \neq |\pi_j|$ for all $i \neq j$

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- Reverse permutation:
  $\eta_n = (-n \ -(n-1) \ \ldots \ -2 \ -1)$

Definitions

- Signed reversal: $\bar{\rho}(i, j)$ with $1 \leq i \leq j \leq n$

$$\begin{array}{rcl} \pi & = & (\pi_1 \ \dots \ \pi_{i-1} \ \underline{\pi_i \ \ \pi_{i+1} \ \dots \ \pi_{j-1} \ \ \pi_j} \ \pi_{j+1} \ \dots \ \pi_n) \\ \pi \cdot \bar{\rho}(i,j) & = & (\pi_1 \ \dots \ \pi_{i-1} \ \underline{-\pi_j \ -\pi_{j-1} \ \dots \ -\pi_{i+1} \ -\pi_i} \ \pi_{j+1} \ \dots \ \pi_n) \end{array}$$

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- Example:

$$\begin{array}{rl} \pi = & (-3 \ \underline{+1 \ -5 \ +2 \ +7} \ -4 \ -3) \\ \pi \cdot \bar{\rho}(2,5) = & (-3 \ \underline{-7 \ -2 \ +5 \ -1} \ -4 \ -3) \end{array}$$

Definitions

- Transposition: $\tau(i, j, k)$ with $1 \leq i < j < k \leq n+1$

$$\begin{aligned} \pi &= (\pi_1 \ldots \pi_{i-1}\ \underline{\pi_i\ \ \pi_{i+1} \ldots \pi_{j-1}}\ \underline{\pi_j\ \ \pi_{j+1} \ldots \pi_{k-1}}\ \pi_k \ldots \pi_n) \\ \pi \cdot \tau(i,j,k) &= (\pi_1 \ldots \pi_{i-1}\ \underline{\pi_j\ \ \pi_{j+1} \ldots \pi_{k-1}}\ \underline{\pi_i\ \ \pi_{i+1} \ldots \pi_{j-1}}\ \pi_k \ldots \pi_n) \end{aligned}$$

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\pi \cdot \tau(i,j,k) & = & (\pi_1 \ldots \pi_{i-1} \; \underline{\pi_j \;\; \pi_{j+1} \ldots \pi_{k-1}} \; \underline{\pi_i \;\; \pi_{i+1} \ldots \pi_{j-1}} \; \pi_k \ldots \pi_n)
\end{array}
$$

- Example:

$$
\begin{array}{rl}
\pi = & (-3 \; \underline{+1 \;\; -5} \; \underline{+2 \;\; +7 \;\; -4} \;\; -3) \\
\pi \cdot \tau(2,4,7) = & (-3 \; \underline{+2 \;\; +7 \;\; -4} \; \underline{+1 \;\; -5} \;\; -3)
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- Distance: $d_\beta(\pi)$, minimum number of operations in β needed to transform π into ι_n

Definitions

Example for SbSigPR:

$$
\begin{array}{rl}
\pi = & (\underline{+3}\ \ -1\ \ +4\ \ -2) \\
\bar{\rho}_p(1) \rightarrow & (\underline{-3\ \ -1}\ \ +4\ \ -2) \\
\bar{\rho}_p(2) \rightarrow & (\underline{+1\ \ +3\ \ +4}\ \ -2) \\
\bar{\rho}_p(3) \rightarrow & (\underline{-4\ \ -3\ \ -1\ \ -2}) \\
\bar{\rho}_p(4) \rightarrow & (\underline{+2}\ \ +1\ \ +3\ \ +4) \\
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$$d_{\bar{\rho}_p}(\pi) = 4$$

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$$(+0\ \textcolor{red}{\blacksquare}\ -3\ \ -2 \blacksquare -4 \blacksquare -5 \blacksquare +1\ \textcolor{red}{\blacksquare}\ +6)$$

- Strip: maximal subsequence of π without breakpoints

Problems

Problem	Best approximation factor	Complexity
SbR	1.375 [6]	NP-hard [7]
SbSigR	-	P [8]
SbT	1.375 [9]	NP-hard [10]
SbRT	≈2.83 [11]	?
SbSigRT	2^* [12]	?
SbPR	2 [13]	NP-hard [14]
SbSigPR	2 [15]	?
SbPT	2 [3]	?
SbPRPT	3 [4]	?

SbSigPRSigSR

Tries to remove one breakpoint with one operation:

1. $\pi_j = -\pi_1 + 1$, $2 \leq j \leq n$: $\bar{\rho}_p(j-1)$

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Tries to remove one breakpoint with two operations:

1. $\pi_j = -\pi_i - 1$, $1 \leq i < j \leq n$: $\bar{\rho}_p(j) \cdot \bar{\rho}_p(j-i)$

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2. $\pi_j = \pi_i + 1$, $0 \leq i+1 < j \leq n$: $\bar{\rho}_p(i) \cdot \bar{\rho}_p(j-1)$
3. $\pi_i = -\pi_j + 1$, $1 \leq i < j \leq n$:
   $\bar{\rho}_s(i) \cdot \bar{\rho}_s(n+1-(j-i))$

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   - at most $b + 2$ reversals sort it
3. $\sigma^2 = (\underbrace{-p_1 \ \dots \ -1}_{\ell_1} \ \underbrace{-p_2 \ \dots \ -(p_1{+}1)}_{\ell_2} \ \dots\dots$
   $\underbrace{-n \ \dots \ -(p_b{+}1)}_{\ell_{b+1}})$

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SbSigPRSigSR

$$d_{\bar{\rho}_p\bar{\rho}_s}(\pi) \leq 2b_{\bar{\rho}_p\bar{\rho}_s}(\pi) + 1 \tag{1}$$

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$$d_{\bar{\rho}_p \bar{\rho}_s}(\pi) \leq 2b_{\bar{\rho}_p \bar{\rho}_s}(\pi) + 1 \quad (1)$$

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$$d_{\bar{\rho_p}\bar{\rho_s}}(\pi) \leq 2b_{\bar{\rho_p}\bar{\rho_s}}(\pi) + 1 \qquad (1)$$

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$$\lim_{b(\pi)\to\infty} \frac{2b(\pi)+1}{b(\pi)} = 2 + \lim_{b(\pi)\to\infty} \frac{1}{b(\pi)} = 2 + \epsilon \qquad (3)$$

SbSigPRPT

If $\pi_1 \neq 1$, tries to remove two breakpoints with one operation:

1. Since $\pi \cdot \tau_p(i,j) = (\underline{\pi_i \ \ldots \ \pi_{j-1}} \ \underline{\pi_1 \ \ldots \ \pi_{i-1}} \ \ldots \ \pi_n)$, we must find $\pi_{j-1} = \pi_1 - 1$ and $\pi_{i-1} = \pi_j - 1$

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   - To maintain our approximation, $\pi_i \neq 1$

SbSigPRPT

If $\pi_1 \neq 1$, tries to remove one breakpoint with one operation by increasing the first strip:

1. Let $\pi = (k{+}1\ k{+}2\ \ldots\ k{+}(i-1)\ \pi_i\ \ldots\ldots)$

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3. If $\pi_{j-1} = k = \pi_1 - 1$ exists, then $\tau_p(i, j)$
4. If $\pi_{j+1} = -\pi_1 + 1$ exists, then $\bar{\rho}_p(j)$, for $1 \leq j \leq n$

SbSigPRPT

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1. It will be removed only when n is sent there
2. Which guarantees that $\pi_1 = 1$ again at most one more time
3. Therefore, it will be possible to remove at least one breakpoint until the end of the sorting
   - using at most two extra operations that do not remove breakpoints

SbSigPRPT

$$d_{\bar{\rho}_p \tau_p}(\pi) \leq b_{\bar{\rho}_p}(\pi) + 2 \quad (4)$$

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SbSigPRPT

$$d_{\bar{\rho}_p \tau_p}(\pi) \leq b_{\bar{\rho}_p}(\pi) + 2 \tag{4}$$

$$d_{\bar{\rho}_p \tau_p}(\pi) \geq \frac{b_{\bar{\rho}_p}(\pi)}{2} \tag{5}$$

$$\lim_{b(\pi)\to\infty} \frac{b(\pi) + 2}{\frac{b(\pi)}{2}} = 2 + \lim_{b(\pi)\to\infty} \frac{4}{b(\pi)} = 2 + \epsilon \tag{6}$$

SbSigPRPTSigSRST

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SbSigPRPTSigSRST

Tries to remove two breakpoints with one operation:

1. if $\pi_{j-1} = \pi_1 - 1$ and $\pi_{i-1} = \pi_j - 1$ exists, $2 \leq i < j \leq n$, then $\tau_p(i, j)$
2. if $\pi_i = \pi_n + 1$ and $\pi_j = \pi_{i-1} + 1$ exists, $2 \leq i < j \leq n$, then $\tau_s(i, j)$
3. neither $\pi_{i-1} = n$ and $\pi_i = 1$ nor $\pi_{j-1} = -1$ and $\pi_j = -n$ can happen

SbSigPRPTSigSRST

Tries to remove one breakpoint with one operation:

1. let $\pi = (k+1\ k+2\ \ldots\ k+(i-1)\ \pi_i\ \ldots\ldots)$, to increase the first strip with a prefix transposition:

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   1. if $\pi_j = k+i = \pi_{i-1}+1$, $j \leq n$, exists, then $\tau_p(i,j)$

SbSigPRPTSigSRST

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2. let $\pi = (\ldots\ldots\ \pi_{j-1}\ k+1\ k+2\ \ldots\ k+x)$, to increase the last strip with a suffix transposition:

SbSigPRPTSigSRST

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2. let $\pi = (\ldots\ldots\ \pi_{j-1}\ k+1\ k+2\ \ldots\ k+x)$, to increase the last strip with a suffix transposition:
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3. again, we cannot separate n and 1 or -1 and $-n$
4. if $\pi_{j+1} = -\pi_1 + 1$, $1 \leq j \leq n-1$, exists, then $\bar{\rho}_p(j)$
5. if $\pi_{i-1} = -\pi_n - 1$, $2 \leq i \leq n$, exists, then $\bar{\rho}_s(i)$

SbSigPRPTSigSRST

Otherwise, π is of one of the five forms:

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SbSigPRPTSigSRST

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4. $(\underline{k+1\ k+2\ \ldots\ n}\ \underline{1\ 2\ \ldots\ k})$

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5. $(\underline{-k\ \ -(k-1)\ \ \ldots\ \ -1}\ \underline{-n\ \ -(n-1)\ \ \ldots\ \ -(k+1)})$

For the last four types, we must apply a prefix transposition to concatenate the first strip with the last one

SbSigPRPTSigSRST

$$d_{\bar{\rho}_p \tau_p \bar{\rho}_s \tau_s}(\pi) \leq b_{\bar{\rho}_p \bar{\rho}_s}(\pi) + 2 \tag{7}$$

SbSigPRPTSigSRST

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SbSigPRPTSigSRST

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$$\lim_{b(\pi) \to \infty} \frac{b(\pi) + 2}{\frac{b(\pi)}{2}} = 2 + \lim_{b(\pi) \to \infty} \frac{4}{b(\pi)} = 2 + \epsilon \tag{9}$$

Results

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Results

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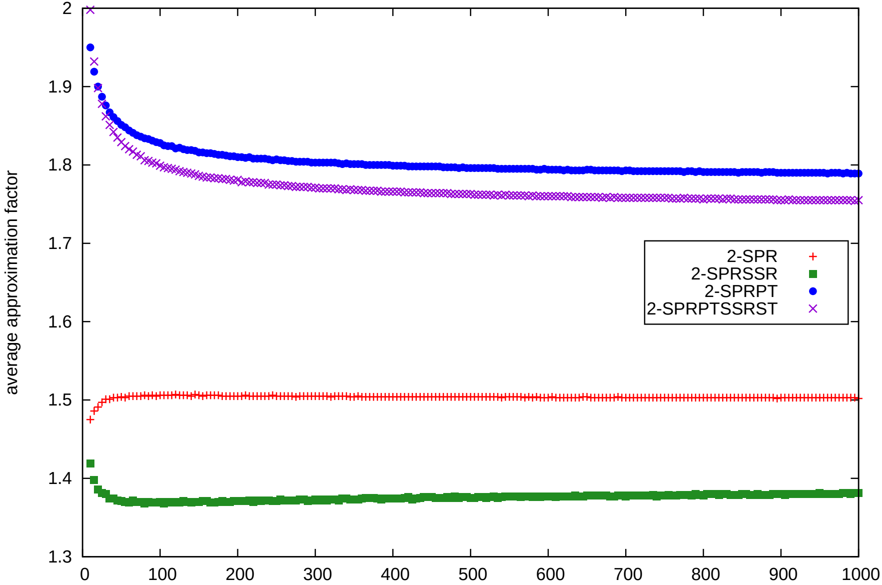
- Approximation factors of the graph are an average between the 10000 of each size n
- Each approximation factor was calculated using the theoretical lower bound

Results



average approximation factor
2
1.9
1.8
1.7
1.6
1.5
1.4
1.3
0
100
200
300
400
500
600
700
800
900
1000
2-SPR
2-SPRSSR
2-SPRPT
2-SPRPTSSRST

Results

Since all algorithms are asymptotic, it is expected to have approximation factors above 2:

- $\textsc{SbSigPRSigSR}$: only when $n = 10$, for one permutation

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- $\textsc{SbSigPRSigSR}$: only when $n = 10$, for one permutation
- $\textsc{SbSigPRPT}$: only when $n \leq 100$, for 0.41% of the permutations (72.13% when $n \leq 20$)

Results

Since all algorithms are asymptotic, it is expected to have approximation factors above 2:

- $\textsc{SbSigPRSigSR}$: only when $n = 10$, for one permutation
- $\textsc{SbSigPRPT}$: only when $n \leq 100$, for 0.41% of the permutations (72.13% when $n \leq 20$)
- $\textsc{SbSigPRPTSigSRST}$: only when $n \leq 105$, for 0.44% of the permutations (76.62% when $n \leq 20$)

Conclusions

- We presented the first results for three sorting problems involving signed prefix and suffix operations: SbSigPRSigSR, SbSigPRPT and SbSigPRPTSigSRST

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Conclusions

- We presented the first results for three sorting problems involving signed prefix and suffix operations: SbSigPRSigSR, SbSigPRPT and SbSigPRPTSigSRST
- All algorithms established a good approximation factor for these problems
- We are currently working on these problems and their unsigned versions

On Sorting of Signed Permutations by Prefix and Suffix Reversals and Transpositions

AlCoB'2014, Tarragona, Spain

Thank you!

July 3, 2014

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